Economic Principles, Spring 2018
Final Exam Solutions

You have two hours to complete this exam. Please answer the following three questions. Be sure to allocate your time in proportion to the points. Always justify your answers by providing a formal proof or a detailed argument. Good luck.

1. **[35 points]** Consider an economy with one consumer and one firm. There are two commodities. The consumer's initial endowment is $(e_1, e_2) = (45, 0)$, where e_j, $j = 1, 2$, denotes the endowment of commodity j.

   The firm produces the second commodity according to the production function $f\left(x_1^f\right) = 2\sqrt{x_1^f}$, where x_1^f denotes the amount of the first commodity purchased by the firm.

   The consumer's utility depends on his bundle (x_1, x_2) and the firm's production x_2^f. In particular, his preferences are given by:

$$u\left(x_1, x_2, x_2^f\right) = \ln x_1 + \ln x_2 - \frac{1}{2}\ln x_2^f$$

   a) Find the Pareto efficient allocation of this economy.

The Pareto efficient allocation is the solution to the following optimization problem:

$$\max_{x_1 \in [0,45]} \ln x_1 + \ln 2\sqrt{45 - x_1} - \frac{1}{2}\ln 2\sqrt{45 - x_1}$$

The solution is $x_1 = 36$. In the Pareto efficient allocation the consumer's bundle is $(36, 6)$ and the firm's production plan is $(-9, 6)$.

1. **b)** Suppose the government introduces a per-unit tax τ on the firm's production (i.e., the firm pays τ on each unit of the second commodity). The government uses the tax revenues to make a transfer T to the consumer. Find the values of τ and T such that the Walrasian equilibrium allocation (of the economy with taxes and transfers) is Pareto efficient. Compute the Walrasian equilibrium prices of the two commodities (normalize $p_1 = 1$).

Let p_2 denote the price of the second commodity. The firm faces the following problem:

$$\max_{x_1 \geq 0} (p_2 - \tau) 2\sqrt{x_1} - x_1$$

The firm's demand of the first commodity is $x_1^f(p_2, \tau) = (p_2 - \tau)^2$. The firm's supply function is $x_2^f(p_2, \tau) = 2(p_2 - \tau)$. Finally, the firm's profits are equal to $\pi(p_2, \tau) = (p_2 - \tau)^2$.

Consider now the consumer's problem:

$$\max_{x_1 \geq 0, x_2 \geq 0} \ln x_1 + \ln x_2 - \frac{1}{2} \ln x_2^f$$
$$\text{s.t. } x_1 + p_2 x_2 \leq 45 + \pi(p_2, \tau) + T$$

The consumer's Marshallian demand is given by:

$$x_1^C(p_2, \tau, T) = \frac{45 + \pi(p_2,\tau) + T}{2} \quad x_2^C(p_2, \tau, T) = \frac{45 + \pi(p_2,\tau) + T}{2p_2}$$

The values of τ, T and p_2 that implement the Pareto efficient allocation are the solutions to the following system of equations:

$$36 = x_1^C(p_2, \tau, T) = \frac{45 + \pi(p_2,\tau) + T}{2} = \frac{45 + (p_2 - \tau)^2 + T}{2}$$
$$6 = x_2^C(p_2, \tau, T) = \frac{45 + (p_2 - \tau)^2 + T}{2p_2}$$
$$6 = x_2^f(p_2, \tau) = 2(p_2 - \tau)$$

We obtain $T = 18$, $\tau = 3$, $p_2 = 6$.

2. **[30 points]** A monopolist faces a consumer who has private information about his valuation of the good. A consumer with low valuation obtains the payoff $3q - x$ if he purchases the quantity q and makes a total payment of x. On the other hand, a consumer with high valuation obtains the payoff $4q - x$ if he purchases the quantity q and makes a total payment of x. Both types of consumers (l and h) obtain a payoff equal to zero if they do not trade with the monopolist. The two types of consumers are equally likely.

   The monopolist's cost of producing quantity q is q^2.

   The monopolist offers a menu with two contracts $((q_l, x_l), (q_h, x_h))$ (one for each type) to maximize his expected profits. Find the optimal menu.

The optimal menu is the solution to the following optimization problem:

$$\max_{((q_l,x_l),(q_h,x_h))} \frac{1}{2}(x_l - q_l^2) + \frac{1}{2}(x_h - q_h^2)$$

$$\text{s.t. } 3q_l - x_l \geq 0 \text{ IR(L)}$$
$$3q_l - x_l \geq 3q_h - x_h \text{ IC(L)}$$
$$4q_h - x_h \geq 0 \text{ IR(H)}$$
$$4q_h - x_h \geq 4q_l - x_l \text{ IC(H)}$$

The constraint IR(L) must bind. To see this, we distinguish between two cases. First, assume that IR(H) is binding. This implies that $3q_h - x_h < 0$. Thus, IC(L) is slack. It is, therefore, possible to increase x_l by a small amount without violating any constraint. Second, assume

that IR(H) is not binding. In this case, it is possible to increase x_l and x_h by the *same* small amount without violating any constraint.
Next, notice that IC(H) and IR(L) imply IR(H). Therefore, we can ignore IC(H). It follows that IC(H) must bind (if not, it is possible to increase x_h without violating the constraints), Thus, we have

$$x_l = 3q_l$$
$$x_h = 4q_h - 4q_l + 3q_l$$

Substituting the values of x_l and x_h in IC(L) we obtain:

$$0 \geq 3q_h - x_h = 3q_h - 4q_h + 4q_l - 3q_l = q_l - q_h$$

We can rewrite the monopolist problem as

$$\max_{q_h \geq q_l \geq 0} \frac{1}{2}\left(3q_l - q_l^2\right) + \frac{1}{2}\left(4q_h - 4q_l + 3q_l - q_h^2\right)$$

The optimal menu is $(q_l, x_l) = (1, 3)$ and $(q_h, x_h) = (2, 7)$.

3. **[35 points]** Consider the Rothschild-Stiglitz model of insurance. There are two groups of individuals. Low risk consumers incur a loss $L = 72$ with probability $\pi_l = \frac{19}{40}$. High risk consumers incur the loss $L = 72$ with probability $\pi_h = \frac{1}{2}$. Low and high risk consumers are identical in every other dimension. Their initial wealth is $W = 100$. Their von Neumann-Morgenstern utility function over positive levels of wealth is given by

$$u(x) = \sqrt{x}$$

Suppose that the fraction α of the high risk consumers is sufficiently large that the equilibrium exists.

Characterize the equilibrium of the model. In particular, let (W_1^i, W_2^i) denote the equilibrium contract of type $i = h, l$, where W_1^i represents the wealth in state 1 (no loss) and W_2^i represents the wealth in state 2 (there is a loss). Compute the values of W_1^h and W_2^h. Write the conditions that pin down the values of W_1^l and W_2^l.

The equilibrium of the model is separating. Furthermore, the contract $\left(W_1^h, W_2^h\right)$ yields zero profits and provides full insurance to the high risk consumers (thus, $W_1^h = W_2^h$). Equating the profits to zero we obtain

$$\frac{1}{2}\left(100 - W_1^h\right) = \frac{1}{2}\left(W_1^h - 28\right)$$

We have $\left(W_1^h, W_2^h\right) = (64, 64)$.

The contract $\left(W_1^l, W_2^l\right)$ is pinned down by the following conditions. First, it yields zero profits. Second, the high risk consumers are indifferent between the contract $\left(W_1^h, W_2^h\right)$ and the contract $\left(W_1^l, W_2^l\right)$. Thus, we have:

$$\tfrac{21}{40}\left(100 - W_1^l\right) = \tfrac{19}{40}\left(W_1^l - 28\right)$$
$$\tfrac{1}{2}\sqrt{W_1^l} + \tfrac{1}{2}\sqrt{W_2^l} = \tfrac{1}{2}\sqrt{W_1^h} + \tfrac{1}{2}\sqrt{W_2^h} = 8$$

The contract offered to the low risk consumers is $\left(W_1^l, W_2^l\right) = (81, 49)$.