Collegio Carlo Alberto

Economic Principles

Solutions to Problem Set 1

Question 1.

Let $\succcurlyeq$ be represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Prove that $u(x)$ is strictly quasiconcave if and only if $\succcurlyeq$ is strictly convex.

"If" part: ("strict convexity of $\succcurlyeq$" $\Rightarrow$" strict quasiconcavity of u").
For $x, y \in \mathbb{R}^n_+$,suppose $y \succcurlyeq x$ and $y \neq x$. Strict convexity of $\succcurlyeq$ implies that

$$\alpha y + (1-\alpha) x \succ x$$

for all $\alpha \in (0,1)$. Since u represents $\succcurlyeq$, this means that

$$u(\alpha y + (1-\alpha) x) > u(x) = \min\{u(x), u(y)\}.$$

Hence, u is strictly quasiconcave.

"Only if" part: ("strict quasiconcavity of u"$\Rightarrow$ "strict convexity of $\succcurlyeq$").
Suppose $z \succcurlyeq x$, $y \succcurlyeq x$, with $z \neq y$. We have to show that

$$\alpha y + (1-\alpha) z \succ x$$

for all $\alpha \in (0,1)$. Without loss of generality, suppose $z \succcurlyeq y$, i.e., $u(z) \geq u(y)$. Strict quasiconcavity of u implies

$$u(\alpha y + (1-\alpha) z) > u(y) \geq u(x).$$

Hence, $\succcurlyeq$ is strictly convex.

Question 2.

To prove that the two functions have the same indifference curves, pick an arbitrary bundle (x_1, x_2). For utility function $u(x) = \sqrt{x_1 x_2}$ the bundles y indifferent to (x_1, x_2) satisfy the equality $\sqrt{x_1 x_2} = \sqrt{y_1 y_2}$ which by applying logarithms to both sides is equivalent to $\log x_1 + \log x_2 = \log y_1 + \log y_2$, but this is the same indifference condition we obtain if we use the utility function $v(x) = \log x_1 + \log x_2$. We conclude that the two utility functions give the same set of bundles indifferent to x. Since x was arbitrary, the two utility functions have the same indifference curves.
Since u_0 was chosen arbitrarily, u and v have the same indifference curves.
For u,

$$MRS = \frac{MU_1}{MU_2} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{\frac{\sqrt{x_2}}{2\sqrt{x_1}}}{\frac{\sqrt{x_1}}{2\sqrt{x_2}}} = \frac{x_2}{x_1}$$

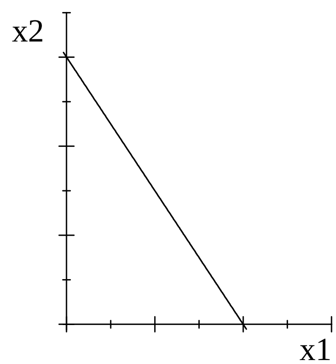
For v,

$$MRS = \frac{MU_1}{MU_2} = \frac{\frac{\partial v}{\partial x_1}}{\frac{\partial v}{\partial x_2}} = \frac{\frac{1}{x_1}}{\frac{1}{x_2}} = \frac{x_2}{x_1}$$

u and v have the same indifference curves and the same MRS because each utility function is a strictly increasing transformation of the other. Specifically, $v(.) = 2\ln(u(.))$.

Question 3.
Graph an indifference curve, and compute the MRS and the Marshallian demand functions for the following utility functions:

a) Perfect substitutes: $u(x_1, x_2) = \alpha x_1 + \beta x_2$, where $\alpha > 0$, $\beta > 0$;



Perfect Substitutes (slope=$-\frac{\alpha}{\beta}$).

$$MRS = \frac{MU_{x_1}}{MU_{x_2}} = \frac{\alpha}{\beta}$$

It is possible to solve the problem graphically . Here we do a little more algebraic solution.

The problem we want to solve is

$$\begin{gathered} \max_{x_1,x_2} \alpha x_{1+}\beta x_2 \\ \text{s.t.} \quad p_1x_1 + p_2x_2 = y \\ x_1 \geq 0, x_2 \geq 0 \end{gathered}$$

The strategy we can use is to solve for x_2 in the budget constraint and substitute it in the objective function, turning it in a maximization problem in one variable. We need some care though, since we need to remember the nonnegativity constraints. From the budget constraint we can write:

$$x_2 = \tfrac{y}{p_2} - \tfrac{p_1}{p_2}x_1$$

However we need to remember that $x_2 \geq 0$, that is x_1 must be contained in the interval $[0, \frac{y}{p_1}]$.
So our original maximization problem is equivalent to the following:

$$\max\ (\alpha - \beta\frac{p_1}{p_2})x_1 + \beta\frac{y}{p_2}$$
$$\text{s.t. } x_1 \in [0, \frac{y}{p_1}]$$

This is a very easy maximization problem, since we are maximazing a straight line over an interval. The slope of the straight line is $(\alpha - \beta\frac{p_1}{p_2})$. Therefore, if the slope is strictly positive (negative), the straight line is strictly increasing (decreasing) , so the point of maximum is at the right (left) endpoint of the interval. If the objective function is constant (the slope is zero) , the consumer will be indifferent tamong all the values in the interval.
We can summarize these observations in our Marshallian demand function:

$$x_1(p,y) = \begin{cases} 0 & \text{if } \frac{\alpha}{\beta} < \frac{p_1}{p_2} \\ [0, \frac{y}{p_1}] & \text{if } \frac{\alpha}{\beta} = \frac{p_1}{p_2} \\ \frac{y}{p_1} & \text{if } \frac{\alpha}{\beta} > \frac{p_1}{p_2} \end{cases}$$

$$x_2(p,y) = \begin{cases} 0 & \text{if } \frac{\alpha}{\beta} > \frac{p_1}{p_2} \\ \frac{y}{p_2} - \frac{p_1}{p_2}x_1(p,y) & \text{if } \frac{\alpha}{\beta} = \frac{p_1}{p_2} \\ \frac{y}{p_2} & \text{if } \frac{\alpha}{\beta} < \frac{p_1}{p_2} \end{cases}$$

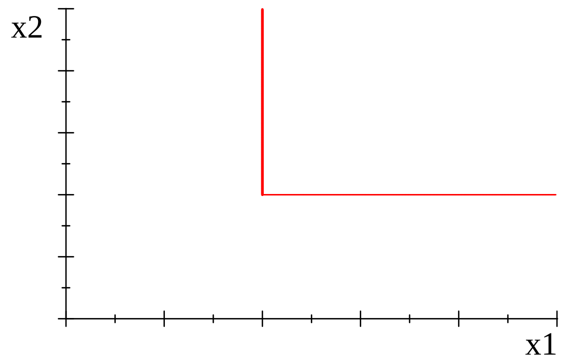
b) The key to solve this problem is the following observation. Suppose the solution satisfies the budget constraint and has $\alpha x_1 \neq \beta x_2$ Without loss of generality suppose $\alpha x_1 > \beta x_2$. The utility of this bundle is βx_2. This utility can be increased if we decrease the consumption of x_1 by a small amount $\varepsilon > 0$, use the money we are saving $(p_1\varepsilon)$ to buy a little more of x_2 $(\frac{p_1}{p_2}\varepsilon)$. The new bundle $(x_1 - \varepsilon, x_2 + \frac{p_1}{p_2}\varepsilon)$ still satisfies the budget constraint with equality, and we can choose ε small enough so that it satisfies the inequality $\alpha(x_1 - \varepsilon) > \beta(x_2 + \frac{p_1}{p_2}\varepsilon)$. The utility of our new bundle is therefore $\beta(x_2 + \frac{p_1}{p_2}\varepsilon) > \beta x_2$ contradicting the hypothesis that our original bundle (x_1, x_2) was optimal. We conclude that the solution to the maximization problem with goods that are perfect complements must satisfy: $ax_1 = \beta x_2$. We can then solve the following sistem:

$$\begin{cases} ax_1 = \beta x_2 \\ p_1x_1 + p_2x_2 = y \end{cases}$$

Hence, the Marshallian demand functions are:

$$x_1(p_1, p_2, y) = \frac{\beta y}{\beta p_1 + \alpha p_2}$$
$$x_2(p_1, p_2, y) = \frac{\alpha y}{\beta p_1 + \alpha p_2}$$

An indifference curve is shown in red in the following graph:



Perfect complements

$MRS = \infty$ when $\beta x_2 > \alpha x_1$;
$MRS = 0$ when $\beta x_2 < \alpha x_1$;
and MRS is not well defined when $\beta x_2 = \alpha x_1$.

Question 4.
(JR 1.21). We have noted that $u(x)$ is invariant to positive monotonic transformation. One common transformation is the *logarithmic transform,* $\ln(u(x))$. Take the logarithmic transform of the Cobb-Douglas utility function; then using that as the utility function, derive the Marshallian demand functions and verify that they are identical to those derived in class.

Cobb-Douglas utility function:

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$$

Taking the logarithmic transformation,

$$v(x_1, x_2) = \ln(u(x_1, x_2)) = \alpha \ln x_1 + \beta \ln x_2$$

To find the Marshallian demand functions, we solve the problem:

$$\begin{gathered} \max \alpha \ln x_1 + \beta \ln x_2 \\ \text{s.t. } p_1 x_1 + p_2 x_2 = y \end{gathered}$$

The Lagrangian for this problem is:

$$L = \alpha \ln x_1 + \beta \ln x_2 + \lambda(y - p_1 x_1 - p_2 x_2)$$

The F.O.C. are:

$$\begin{aligned} \frac{\alpha}{x_1} &= \lambda p_1 \\ \frac{\beta}{x_2} &= \lambda p_2 \\ p_1 x_1 + p_2 x_2 &= y \end{aligned}$$

Taking the ratio of the first two equations gives:

$$\frac{\alpha x_1}{\beta x_2} = \frac{p_1}{p_2}$$

Together with the budget constraint, we can solve for the optimal choice of x_1 and x_2.

$$\frac{\alpha x_1}{\beta \left(y/p_2 - p_1 x_1/p_2 \right)} = \frac{p_1}{p_2} \Rightarrow x_1 = \frac{\alpha}{\alpha + \beta} \cdot \frac{y}{p_1}$$

Substitute this into the budget constraint, we get

$$x_2 = \frac{\beta}{\alpha + \beta} \cdot \frac{y}{p_2}$$

Hence, the Marshallian demand functions are the same as those we derived in class.

Question 5.
(JR 1.27). A consumer of two goods faces positive prices and has a positive income. Her utility function is

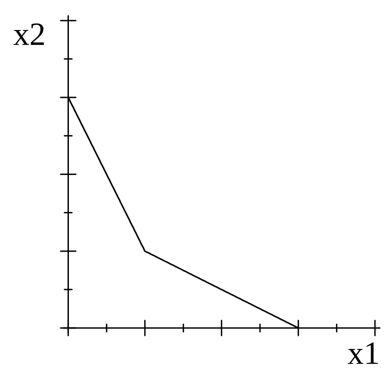
$$u\left(x_1, x_2\right) = \max\left\{a x_1, a x_2\right\} + \min\left\{x_1, x_2\right\}, \quad \text{where } 0 < a < 1.$$

Derive the Marshallian demand functions.

We can express the utility function in the following way:

$$\begin{aligned} u(x_1, x_2) &= \max\{a x_1, a x_2\} + \min\{x_1, x_2\} \\ &= \begin{cases} a x_1 + x_2 & if \quad x_1 \geq x_2 \\ x_1 + a x_2 & if \quad x_1 \leq x_2 \end{cases} \end{aligned}$$

Graphically, an indifference curve looks like this:



Indifference Curve.

Let's solve the maximization problem. We can distinguish several cases, depending on the relationship between the price ratio (i.e. the slope of the budget constraint) and the MRS (i.e. the slope of indifference curves).

- $\frac{p_1}{p_2} < a < \frac{1}{a}$.

  In this case the budget constraint is flatter than both the MRS above and below the 45-degree line. So the consumer will buy only good 1.

  $x_1(p, y) = \frac{y}{p_1}$

  $x_2(p, y) = 0$

- $\frac{p_1}{p_2} = a < \frac{1}{a}$.

  In this case the consumer is indifferent among all the bundles on the budget set and *below* the 45-degree line.

  $x_1(p, y) \in [\frac{y}{p_1+p_2}, \frac{y}{p_1}]$

  $x_2(p, y) = \frac{y}{p_2} - \frac{p_1}{p_2}x_1(p, y)$

- $a < \frac{p_1}{p_2} < \frac{1}{a}$.

  In this case the slope of the budget constraint is between the two MRS and the maximal point is at the kink, that is where $x_1 = x_2$ By solving keeping in mind that the solution must satisfy the budget constraint we get

  $x_1(p_1,p_2,y) = \frac{y}{p_1+p_2}$ $x_2(p_1,p_2,y) = \frac{y}{p_1+p_2}$

- $a < \frac{p_1}{p_2} = \frac{1}{a}$.

  In this case the consumer is indifferent between all the bundles that are on the budget line and *above* the 45-degree line.

  $x_1(p, y) \in [0, \frac{y}{p_1+p_2}]$

  $x_2(p, y) = \frac{y}{p_2} - \frac{p_1}{p_2}x_1(p, y)$

- $a < \frac{1}{a} < \frac{p_1}{p_2}$

  In this case the budget constraint is steeper than both MRS, therefore the consumer will consume only good 2.

  $x_1(p, y) = 0$

  $x_2(p, y) = \frac{y}{p_2}$

Question 6

Consider the following monotonic transformation of the $u(.)$:

$$v(.) = (u(.))^2 = x_1 + 2x_2 + 3x_3.$$

The three goods are perfect substitutes for each other and the consumer will choose the good that gives the highest $\frac{MU_i}{p_i}$.

The Marshallian demand functions are as follows:

if $p_1 < \min\{\frac{p_2}{2}, \frac{p_3}{3}\}$, $x_1(\mathbf{p}, y) = \frac{y}{p_1}$, $x_2(\mathbf{p}, y) = 0$, $x_3(\mathbf{p}, y) = 0$;
if $\frac{p_2}{2} < \min\{p_1, \frac{p_3}{3}\}$, $x_1(\mathbf{p}, y) = 0$, $x_2(\mathbf{p}, y) = \frac{y}{p_2}$, $x_3(\mathbf{p}, y) = 0$;
if $\frac{p_3}{3} < \min\{p_1, \frac{p_2}{2}\}$, $x_1(\mathbf{p}, y) = 0$, $x_2(\mathbf{p}, y) = 0$, $x_3(\mathbf{p}, y) = \frac{y}{p_3}$;
if $p_1 = \frac{p_2}{2} < \frac{p_3}{3}$, $x_3(\mathbf{p}, y) = 0$, $x_1(\mathbf{p}, y) \geq 0$, $x_2(\mathbf{p}, y) \geq$ and $p_1 x_1(\mathbf{p}, y) + p_2 x_2(\mathbf{p}, y) = y$;
if $p_1 = \frac{p_3}{3} < \frac{p_2}{2}$, $x_2(\mathbf{p}, y) = 0$, $x_1(\mathbf{p}, y) \geq 0$, $x_3(\mathbf{p}, y) \geq$ and $p_1 x_1(\mathbf{p}, y) + p_3 x_3(\mathbf{p}, y) = y$;
if $\frac{p_2}{2} = \frac{p_3}{3} < p_1$, $x_1(\mathbf{p}, y) = 0$, $x_2(\mathbf{p}, y) \geq 0$, $x_3(\mathbf{p}, y) \geq$ and $p_2 x_2(\mathbf{p}, y) + p_3 x_3(\mathbf{p}, y) = y$;
if $p_1 = \frac{p_2}{2} = \frac{p_3}{3}$, $x_1(\mathbf{p}, y) \geq 0$, $x_2(\mathbf{p}, y) \geq 0$, $x_3(\mathbf{p}, y) \geq$ and $p_1 x_1(\mathbf{p}, y) + p_2 x_2(\mathbf{p}, y) + p_3 x_3(\mathbf{p}, y) = y$.

When $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, we have $\frac{p_2}{2} < \min\{p_1, \frac{p_3}{3}\}$ and therefore $x_1(\mathbf{p}, y) = 0$, $x_2(\mathbf{p}, y) = \frac{y}{3}$, $x_3(\mathbf{p}, y) = 0$.
To achieve utility level 6, she needs income y such that

$$\sqrt{2 \times \frac{y}{3}} = 6 \Longrightarrow y = 54$$