Collegio Carlo Alberto

Economic Principles

Solutions to Problem Set 2

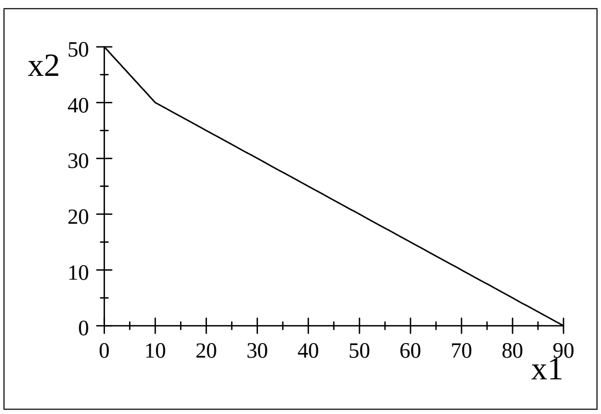
Question 1.
Given the conditions, Bob's budget constraint is equal to:

$$\begin{cases} 2x_1 + 2x_2 = 100 & \text{if } x_1 \leq 10 \\ (x_1 - 10) + 2x_2 = 100 - 2 \times 10 & \text{if } \ x_1 > 10 \end{cases}$$

i.e.,

$$\begin{cases} 2x_1 + 2x_2 = 100 & \text{if } x_1 \leq 10 \\ x_1 + 2x_2 = 90 & \text{if } \ x_1 > 10 \end{cases}$$

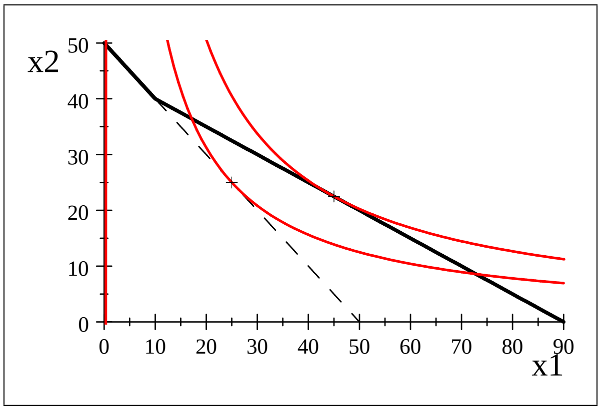
Graphically, the budget constraint looks as follows:



Now we want to find Bob's optimal consumption plan. Since the utility function is Cobb-Douglas, we know that if the budget constraint is

$$2x_1 + 2x_2 = 100$$

then Bob would consume $x_1 = \frac{100}{2\times 2} = 25$ and $x_2 = \frac{100}{2\times 2} = 25$. Since $x_1 = 25 > 10$, this is in the interior of Bob's real budget constraint. We know that it's not optimal. The following picture illustrates the point. The thick line is Bob's actual budget constraint, while the thin line is the budget constraint that would occur if there was no discount on the price of ice cream.



For budget constraint $x_1 + 2x_2 = 90$, Bob would consume $x_1 = \frac{90}{2\times 1} = 45$ and $x_2 = \frac{90}{2\times 2} = 22.5$. Since $x_1 = 45 > 10$, we conclude that Bob's optimal consumption plan is $(45, 22.5)$.

Question 2.

(a) Marshallian demand functions.

To simplify the derivation of the Marshallian demand function, it is convenient to take the logarithmic transformation of the utility function (of course, this will not change our results). Let $v\left(x_1, ..., x_n\right)$ be defined by:

$$v(x_1, x_2, ..., x_n) = \ln(u(x_1, x_2, ..., x_n)) = \ln(A\prod_{i=1}^{n} x_i^{\alpha_i}) = \ln A + \sum_{i=1}^{n} \alpha_i \ln x_i.$$

Hence, we have the Lagrangian

$$\pounds = \ln A + \sum_{i=1}^{n} \alpha_i \ln x_i + \lambda(y - \sum_{i=i}^{n} p_i x_i)$$

F.O.C.

$$\begin{aligned} \frac{\alpha_i}{x_i} - \lambda p_i &= 0 \qquad \text{for } i = 1, 2, ..., n \\ \sum_{i=1}^{n} p_i x_i &= y \end{aligned}$$

F.O.C. implies,

$$x_i = \frac{\alpha_i}{\lambda p_i} \tag{1}$$

Since $\sum_{i=1}^{n} p_i x_i = y$, substituting Equation (1) into the budget constraint, we have

$$\sum_{i=1}^{n} p_i \frac{\alpha_i}{\lambda p_i} = y$$

Since $\sum_{i=1}^{n} \alpha_i = 1$, we have

$$\lambda = \frac{1}{y} \tag{2}$$

Substitute Equation (2) into Equation (1), we have the Marshallian demand functions

$$x_i(\mathbf{p}, y) = \frac{\alpha_i}{\frac{1}{y} \cdot p_i} = \frac{a_i y}{p_i} \qquad \text{for } i = 1, 2, ..., n$$

(b) Substituting Marshallian demand into the utility function we obtain the indirect utility function:

$$V(\mathbf{p}, y) = A \prod_{i=1}^{n} (\frac{\alpha_i y}{p_i})^{\alpha_i} = A \prod_{i=1}^{n} (\frac{\alpha_i}{p_i})^{\alpha_i} \prod_{i=1}^{n} y^{\alpha_i} = A \prod_{i=1}^{n} (\frac{\alpha_i}{p_i})^{\alpha_i} y$$

(c) Expenditure function.
We could solve the expenditure minimization problem, find Hicksian demand and compute the expenditure function. However, since we already know the indirect utility function, the easiest way to find the expenditure function is through the duality relationship

$$V(\mathbf{p}, e(\mathbf{p}, u)) = u$$

Hence,

$$A \prod_{i=1}^{n} (\frac{\alpha_i}{p_i})^{\alpha_i} e(\mathbf{p}, u) = u$$

Solving for $e(\mathbf{p}, u)$, we have

$$e(\mathbf{p}, u) = \frac{u}{A} \prod_{i=1}^{n} (\frac{p_i}{\alpha_i})^{\alpha_i}$$

(d) Hicksian demand functions.

$$h_i(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_i} = \frac{\alpha_i \frac{u}{A} \prod_{j=1}^{n} (\frac{p_j}{\alpha_j})^{a_j}}{p_i} = \frac{u}{A} \cdot \frac{\alpha_i \prod_{j=1}^{n} (\frac{p_j}{\alpha_j})^{a_j}}{p_i}.$$

Question 3.
We have the following Lagrangian

$$L = \ln x_1 + x_2 + \lambda(y - p_1 x_1 - p_2 x_2)$$

F.O.C. implies that

$$\begin{aligned} \tfrac{1}{x_1} &= \lambda p_1 \\ 1 &= \lambda p_2 \end{aligned}$$

Hence, the Marshallian demand functions are

$$\begin{aligned} x_1(p_1, p_2, y) &= \tfrac{1}{\frac{1}{p_2} \cdot p_1} = \tfrac{p_2}{p_1} \\ x_2(p_1, p_{2,} y) &= \tfrac{y - p_1 x_1}{p_2} = \tfrac{y - p_2}{p_2} \end{aligned}$$

Cross-price effects:

$$\begin{aligned}\frac{\partial x_1}{\partial p_2} &= \frac{1}{p_1} \\ \frac{\partial x_2}{\partial p_1} &= 0\end{aligned}$$

Hence, the cross-price effects do not coincide.

Question 4

First note that the First-Order Conditions from utility maximization are:

$$g_1^{'}(x_1) = \lambda p_1 \tag{3}$$

$$g_2^{'}(x_2) = \lambda p_2 \tag{4}$$

$$g_3^{'}(x_3) = \lambda p_3 \tag{5}$$

Equations (3) and (4) give us $p_1 g_2^{'}(x_2) = p_2 g_1^{'}(x_1)$. Note that this equation is identically true (i.e. holds for each p, y) at the optimal point (Marshallian demand) $(x_1(p, y), x_2(p, y))$, so we can differentiate it partially with respect to y to obtain

$$p_1 g_2^{''}(x_2(\mathbf{p}, y)) \frac{\partial x_2(\mathbf{p}, y)}{\partial y} = p_2 g_1^{''}(x_1(\mathbf{p}, y)) \frac{\partial x_1(\mathbf{p}, y)}{\partial y}$$

We know prices are strictly positive and that both $g_1^{''}(.)$ and $g_2^{''}(.)$ are strictly negative by concavity, so it must be that both $\frac{\partial x_2(\mathbf{p},y)}{\partial y}$and $\frac{\partial x_1(\mathbf{p},y)}{\partial y}$ have the same sign. By a similar argument, it follows that $\frac{\partial x_2(\mathbf{p},y)}{\partial y}$ and$\frac{\partial x_3(\mathbf{p},y)}{\partial y}$ have the same sign. In other words, all three income effects have the same sign, i.e. all three goods must be inferior or normal.
However, it must be that at least one of the good is normal. In fact, we know that Marshallian demand satisfies budget balancedness, $\sum_i p_i x_i(\mathbf{p}, y) = y$. Differentiating this expression with respect to y gives:

$$\sum_i p_i \frac{\partial x_i(\mathbf{p}, y)}{\partial y} = 1 > 0$$

Since prices are positive, we must have at least one positive $\frac{\partial x_i(\mathbf{p},y)}{\partial y}$.Therefore, they are all positive, so all three goods are normal.
Question 5.
(a) Marshallian demand functions

$$L(x_1, x_2, \lambda) = (x_1)^{\frac{1}{2}} + (x_2)^{\frac{1}{2}} + \lambda(y - p_1 x_1 - p_2 x_2)$$

F.O.C.

$$
\begin{aligned}
\frac{1}{2}(x_1)^{-\frac{1}{2}} &= \lambda p_1 \\
\frac{1}{2}(x_2)^{-\frac{1}{2}} &= \lambda p_2 \\
p_1 x_1 + p_2 x_2 &= y
\end{aligned}
$$

Dividing the first equation by the second equation is always a good trick to eliminate the Lagrange multiplier:

$$
\frac{\frac{1}{2}(x_1)^{-\frac{1}{2}}}{\frac{1}{2}(x_2)^{-\frac{1}{2}}} = \frac{p_1}{p_2} \Rightarrow \frac{x_2}{x_1} = \frac{(p_1)^2}{(p_2)^2} \Rightarrow x_2 = x_1 \cdot \frac{(p_1)^2}{(p_2)^2}
$$

Substituting into the budget constraint, we have

$$
\begin{aligned}
x_1(p_{1,}p_{2,}y) &= \frac{y \cdot p_2}{p_1 p_2 + (p_1)^2} \\
x_2(p_1, p_2, y) &= \frac{y \cdot p_1}{p_1 p_2 + (p_2)^2}
\end{aligned}
$$

(b) The substitution term in the Slutsky equation:

$$
\begin{aligned}
\frac{\partial h_1(p_1, p_2, y)}{\partial p_2} &= \frac{\partial x_1(p_1, p_2, y)}{\partial p_2} + \frac{\partial x_1(p_1, p_2, y)}{\partial y} \cdot x_2(p_1, p_2, y) \\
&= \frac{y(p_1 p_2 + (p_1)^2) - y p_1 p_2}{(p_1 p_2 + p_1^2)^2} + \frac{p_2}{p_1 p_2 + p_1^2} \cdot \frac{p_1 y}{p_1 p_2 + p_2^2} \\
&= \frac{y p_1^2 (p_1 p_2 + p_2^2 + p_1 p_2 + p_2^2)}{(p_1 p_2 + p_1^2)^2 (p_1 p_2 + p_2^2)} = \frac{2 y p_1^2}{(p_1 p_2 + p_1^2)^2} \\
&= \frac{2y}{(p_1 + p_2)^2}
\end{aligned}
$$

(c) Goods 1 and 2 are gross substitutes if $\frac{\partial x_1(p_1,p_2,y)}{\partial p_2} > 0$ (notice, this is the derivative of the *Marshallian demand*). Since

$$
\begin{aligned}
\frac{\partial x_1(p_1, p_2, y)}{\partial p_2} &= \frac{y}{(p_1 + p_2)^2} > 0 \\
\frac{\partial x_2(p_1, p_2, y)}{\partial p_1} &= \frac{y}{(p_1 + p_2)^2} > 0\ ,
\end{aligned}
$$

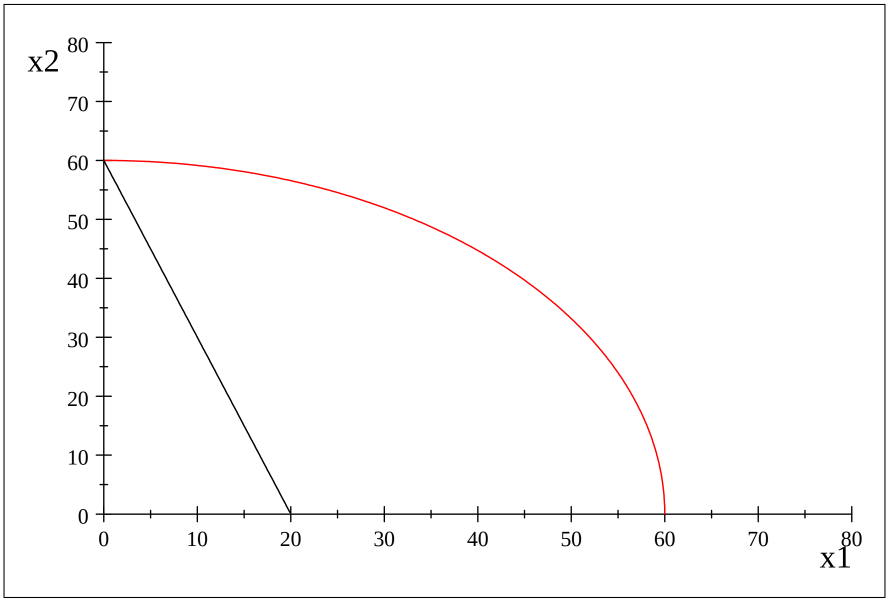
x_1 and x_2 are gross substitutes.

Question 6

Given John's utility function (note that $MU(x_1)$ is increasing in x_1 and $MU(x_2)$ is increasing in x_2), his optimal choice is always a corner solution.

When $p_1 = 15$, $p_2 = 5$, John's optimal choice is to spend all his income on x_2. Hence, in period 0, John's utility level is $\sqrt{0^2 + (\frac{300}{5})^2} = 60$

The following graph illustrates



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In period 1, $p_1' = 20$, $p_2' = 5$. Again, John's optimal choice is to spend all his income on x_2. Since p_2 doesn't change, the income which allows John to obtain in period 1 the same level of utility as in period 0 is \$300.