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Economic Principles
Solutions to Problem Set 3

Question 1
The WARP requires that, if a bundle x_0 is chosen when another bundle x_1 is available, then when this new bundle x_1 is itself chosen, x_0 must not be available. Assuming budget balancedness, $w^i = p^i x^i$
(a) $w^0 = p^0x^0 = 10$, $p^0x^1 = 6$, $w^1 = p^1x^1 = 14$, $p^1x^0 = 22$
Since $w^0 > p^0x^1$ and $w^1 < p^1x^0$, WARP is satisfied.

(b) $w^0 = p^0x^0 = 40$, $p^0x^1 = 32$, $w^1 = p^1x^1 = 44$, $p^1x^0 = 55$
Since $w^0 > p^0x^1$ and $w^1 < p^1x^0$, WARP is satisfied.

(c) $w^0 = p^0x^0 = 5$, $p^0x^1 = 5$, $w^1 = p^1x^1 = 6$, $p^1x^0 = 8$
Since $w^0 = p^0x^1$ and $w^1 < p^1x^0$, WARP is satisfied.

(d) $p^0x^0 = 100$, $p^0x^1 = 60$, $p^1x^1 = 74$, $p^1x^0 = 110$
Since $w^0 > p^0x^1$ and $w^1 < p^1x^0$, WARP is satisfied.

Question 2
Denote by C_i the lottery that gives outcome c_i with certainty. Order the C_i such that $C_1 \succcurlyeq C_2 \succcurlyeq C_3$. Now note that any lottery $L = (p_1, p_2, p_3)$ can be written as $p_1C_1+p_2C_2+p_3C_3$. The idea of our proof is as follows. If a lottery L_1 is as good as another lottery L_2 then independence implies that it is also at least as good as any convex combination of L_1 and L_2. The generalization of this result is that if a lottery L_1 is at least as good as each of the lotteries $L_2, L_3, ..., L_n$ then it is at least as good as any combination of $L_1, L_2, ..., L_n$. Since C_1 is at least as good as C_2 and C_3, it is also at least as good as any lottery L since every lottery can be written as a combination of C_1, C_2 and C_3. However, actually establishing this result requires a few steps since the original definition of independence is in terms of pairs of lotteries.

Since $C_2 \succcurlyeq C_3$, by independence axiom (IA), mixing C_2 and C_3 with C_2

$$\alpha C_2 + (1-\alpha)C_2 \succcurlyeq \alpha C_3 + (1-\alpha)C_2, \ \forall \alpha \in [0,1],$$

i.e.,

$$C_2 \succcurlyeq \alpha C_3 + (1-\alpha)C_2, \ \forall \alpha \in [0,1].$$

Since $C_1 \succcurlyeq C_2$, by transitivity, $C_1 \succcurlyeq \alpha C_3 + (1-\alpha)C_2$.
Again, by IA, mixing C_1 and $\alpha C_3 + (1-\alpha)C_2$ with C_1

$$\beta C_1 + (1-\beta)C_1 \succcurlyeq \beta(\alpha C_3 + (1-\alpha)C_2) + (1-\beta)C_1, \ \forall \alpha, \beta \in [0,1].$$

Hence,

$$\begin{aligned} C_1 &\succcurlyeq \beta(\alpha C_3 + (1-\alpha)C_2) + (1-\beta)C_1 \\ &= (1-\beta)C_1 + (1-\alpha)\beta C_2 + \alpha\beta C_3 \\ &= ((1-\beta), (1-\alpha)\beta, \alpha\beta) \end{aligned}$$

Let $\beta = 1 - p_1, \alpha = \frac{1-p_1-p_2}{1-p_1}$,
we have $C_1 \succcurlyeq L$ where $L = (p_1, p_2, p_3)$.
Hence, $C_1 \succcurlyeq L$ for any $L \in \mathcal{L}$

Similarly, we can show that $L \succcurlyeq C_3$, for any $L = (p_1, p_2, p_3)$ in $\mathcal{L}$.

Question 3

We can prove the statement either directly using linearity of U or using the definition of VNM utility function. Since U is VNM iff U is linear, the two proofs are equivalent.
Suppose U has the expected utility form (VNM form),

$$U(L) = \sum_{i=1}^{n} p_i u_i \quad \text{where } L = (p_1, p_2, ..., p_n)$$

To show that $\succcurlyeq$ satisfies the independence axiom, we need to show that for lotteries L, L' and L'',

$$L \succcurlyeq L' \iff \alpha L + (1-\alpha)L'' \succcurlyeq \alpha L' + (1-\alpha)L'' \qquad \forall L, L', L'' \in \mathcal{L} \text{ and } \alpha \in (0,1)$$

Suppose $L = (p_1, p_2, ..., p_n), L' = (p'_1, p'_2, ..., p'_n), L'' = (p''_1, p''_2, ..., p''_n)$.
Then

$$\begin{gathered} L \succcurlyeq L' \iff U(L) \geq U(L') \\ \iff \textstyle\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} p'_i u_i \\ \iff \alpha \textstyle\sum_{i=1}^{n} p_i u_i \geq \alpha \sum_{i=1}^{n} p'_i u_i \quad \forall \alpha \in (0,1) \\ \iff \alpha \textstyle\sum_{i=1}^{n} p_i u_i + (1-\alpha)\sum_{i=1}^{n} p''_i u_i \geq \alpha \sum_{i=1}^{n} p'_i u_i + (1-\alpha)\sum_{i=1}^{n} p''_i u_i \\ \iff \textstyle\sum_{i=1}^{n} (\alpha p_i + (1-\alpha)p''_i)\, u_i \geq \sum_{\iota=1}^{n} (\alpha p'_i + (1-\alpha)\, p''_i)\, u_i \\ \iff U(aL + (1-\alpha)L'') \geq U(\alpha L' + (1-\alpha)L'') \\ \iff \alpha L + (1-\alpha)L'' \succcurlyeq \alpha L' + (1-\alpha)L'' \end{gathered}$$

Hence, $\succcurlyeq$ satisfies the independence axiom if U has the expected utility form.
Using linearity the proof goes as follow. Suppose U is linear. Then

$$\begin{gathered} L \succcurlyeq L' \iff U(L) \geq U(L') \\ \iff \alpha U(L) \geq \alpha U(L') \\ \iff \alpha U(L) + (1-\alpha)U(L'') \geq \alpha U(L') + (1-\alpha)U(L'') \\ \iff U(\alpha L + (1-\alpha)L'') \geq U(\alpha L + (1-\alpha)L'') \text{ (by linearity)} \\ \iff \alpha L + (1-\alpha)L'' \succcurlyeq \alpha L' + (1-\alpha)L'' \end{gathered}$$

Question 4

Let $c_1 = \$0$, $c_2 = \$5000$, $c_3 = \$30,000$, then

$L_1 = (0, 1, 0)$, $L_2 = (\frac{1}{100}, \frac{89}{100}, \frac{10}{100})$, $L_3 = (\frac{89}{100}, \frac{11}{100}, 0)$, $L_4 = (\frac{9}{10}, 0, \frac{1}{10})$
Let's assume, as usual, that $\succcurlyeq$ is complete, transitive and continuous. If IA is satisfied, then there exists a vN-M utility function that represents $\succcurlyeq$ where $U(L) = \sum_{i=1}^{3} p_i u_i$. Then,

$$\begin{aligned}
L_1 &\succcurlyeq L_2 \Rightarrow U(L_1) \geq U(L_2) \\
&\Rightarrow u_2 \geq \frac{1}{100}u_1 + \frac{89}{100}u_2 + \frac{10}{100}u_3 \\
&\Rightarrow \frac{11}{100}u_2 \geq \frac{1}{100}u_1 + \frac{10}{100}u_3 \\
&\Rightarrow \frac{89}{100}u_1 + \frac{11}{100}u_2 \geq \frac{89}{100}u_1 + \frac{1}{100}u_1 + \frac{10}{100}u_3 \text{ (adding } \frac{89}{100} \text{ to both sides)} \\
&\Rightarrow \frac{89}{100}u_1 + \frac{11}{100}u_2 \geq \frac{90}{100}u_1 + \frac{10}{100}u_3 \\
&\Rightarrow U(L_3) \geq U(L_4) \\
&\Rightarrow L_3 \succcurlyeq L_4
\end{aligned}$$

The preferences $L_1 \succcurlyeq L_2$ and $L_4 \succcurlyeq L_3$ do not violate the IA because we can have indifference between L_1 and L_2 and between L_3 and L_4.
However, if the preference relation is strict, i.e., $L_1 \succ L_2$ and $L_4 \succ L_3$, the conclusion is different. As can be seen from above, if IA is satisfied, then $L_1 \succ L_2$ implies that $L_3 \succ L_4$. Hence, the preferences $L_1 \succ L_2$ and $L_4 \succ L_3$ are inconsistent with the IA.

Question 5

$$\begin{aligned}
L_1 &\sim L_2 \Longrightarrow U(L_1) = U(L_2) \Longrightarrow \frac{1}{2}u_2 + \frac{1}{2}u_3 = \frac{1}{2} \times 4 = 2 \\
L_3 &\sim L_4 \Longrightarrow U(L_3) = U(L_4) \Longrightarrow \frac{1}{3}u_2 + \frac{1}{3}u_3 + \frac{1}{3} \times 4 = \frac{1}{6}u_2 + \frac{5}{6}u_3
\end{aligned}$$

Solving

$$\begin{cases} \frac{1}{2}u_2 + \frac{1}{2}u_3 = 2 \\ \frac{1}{3}u_2 + \frac{1}{3}u_3 + \frac{1}{3} \times 4 = \frac{1}{6}u_2 + \frac{5}{6}u_3 \end{cases},$$

we get

$$\begin{cases} u_2 = 1 \\ u_3 = 3 \end{cases}.$$

Question 6

$$U(p_1, p_2) = 3(p_1)^2 + 4\,(p_2)^2 + 4\sqrt{3}p_1p_2 = (\sqrt{3}p_1 + 2p_2)^2$$

To find the optimal lottery, we solve the following problem:

$$\begin{aligned}
&\max 3(p_1)^2 + 4\,(p_2)^2 + 4\sqrt{3}p_1p_2 \\
&\text{s.t.} \quad p_1 + p_2 = 1 \text{ and } p_1, p_2 \geq 0
\end{aligned}$$

We can substitute one of the probabilities in the objective function using the constraint $p_1+p_2=1$. We then find that the obective function is strictly increasing in p_2 (and therefore, since $p_1 = 1-p_2$, decreasing in p_2) . Therefore, the solution is $p_1^* = 0$ and $p_2^* = 1$.
Since $U(p_1,p_2) = (\sqrt{3}p_1+2p_2)^2$ represents $\succcurlyeq$, $V = \sqrt{U} = \sqrt{3}p_1 + 2p_2$ also represents $\succcurlyeq$ .This is because the order of lotteries is not changed if we transform U with a strictly increasing transformation f. That is $U(L) \geq U(L') \iff f(U(L)) \geq f(U(L'))$
Since V has the expected utility form, we know from question 3 that $\succcurlyeq$ satisfies the independence axiom.
Remark: it is important not to get confused on what the Expected Utility Theorem (EUT) says. If the preference relation is rational and continuous, and if IA is satisfied, the EUT ensures the existence of *a* utility function with the VNM property (i.e. linear in probabilities). Starting from this representation, if we take strictly increasing transformations that are *not* linear, we end up with a new utility function that still represents $\succcurlyeq$ but that is not of VNM form. However, if we take strictly positive affine transformations (the result we proved in class) the new utility function will still represent $\succcurlyeq$ (since the transformation is strictly increasing) and moreover will be of VNM form.