Collegio Carlo Alberto

Economic Principles
Solutions to Problem Set 4

**Question 1**
The investor's problem is

$$\max_{\alpha\in[0,1]} \sum_{i=1}^{n} \pi_i u(w\cdot(1-\alpha)\cdot r + w\cdot\alpha\cdot x_i)$$

Let $w_i = w\cdot(1-\alpha)\cdot r + w\cdot\alpha\cdot x_i$.
For simplicity, let's assume that we have an interior solution, then F.O.C. implies that

$$\sum_{i=1}^{n} \pi_i u'(w_i^*)(w\cdot x_i - w\cdot r) = 0 \Longrightarrow \sum_{i=1}^{n} \pi_i u'(w_i^*)(x_i - r) = 0 \text{ where } w_i^* = w(1-\alpha^*)r + w\alpha^* x_i$$

By the implicit function theorem, we have

$$\frac{d\alpha^*}{dw} = -\frac{d\left(\sum_{i=1}^{n}\pi_i u'(w_i^*)(x_i-r)\right)}{dw} \Big/ \frac{d\left(\sum_{i=1}^{n}\pi_i u'(w_i^*)(x_i-r)\right)}{d\alpha^*}$$

For an investor with constant relative risk aversion,

$$-\frac{u''(w)w}{u'(w)} = r_r \Longrightarrow -u''(w)w = r_r\cdot u'(w)$$

We have

$$\begin{aligned}
\frac{d\left(\sum_{i=1}^{n}\pi_i u'(w_i^*)(x_i-r)\right)}{dw} &= \sum_{i=1}^{n}\left(\pi_i u''(w_i^*)\,(x_i-r)\,((1-\alpha^*)r+\alpha^* x_i)\right) \\
&= \sum_{i=1}^{n}\left(\pi_i u''(w^*)\,(x_i-r)\,\frac{w_i^*}{w}\right) \\
&= \frac{-r_r}{w}\sum_{i=1}^{n}\left(\pi_i u'(w_i)(x_i-r)\right) \\
&= 0 \quad \text{by F.O.C.}
\end{aligned}$$

Hence,

$$\frac{d\alpha^*}{dw} = -\frac{d\left(\sum_{i=1}^{n}\pi_i u'(w_i^*)(x_i-r)\right)}{dw} \Big/ \frac{d\left(\sum_{i=1}^{n}\pi_i u'(w_i^*)(x_i-r)\right)}{d\alpha^*} = 0.$$

Remark: it is important not to get confused between investing an amount of money $k$ in the risky asset and investing a fraction $\alpha$ of wealth in the risky asset. In the first case the agent problem is

$$\max_{0\leq k\leq w} \sum_{i=1}^{n} \pi_i u((kx_i + (w-k)r)$$

The second case is the one solved in this problem. The difference between the two cases lies in the fact that with $CARA$ the agent will invest a fixed (independent of initial wealth) *amount* of money in the risky asset, while with $CRRA$ the investor will invest a fixed $fraction$ of initial wealth in the risky asset. Obviously, if the agent invests a constant fraction of wealth, the amount of money invested in the risky asset will be increasing in initial wealth.

**Question 2**

If the investor is risk-neutral, he wants to choose $\alpha$ to maximize his expected income

$$w(1-\alpha)r + \sum_{i=1}^{n}(w\cdot\alpha\cdot x_i\cdot\pi_i) = w\cdot r + w\alpha\left(\sum_{i=1}^{n}\pi_i x_i - r\right)$$

Hence,

when $\sum_{i=1}^{n}\pi_i x_i > r$, $\alpha^* = 1$;

when $\sum_{i=1}^{n}\pi_i x_i < r$, $\alpha^* = 0$;

when $\sum_{i=1}^{n}\pi_i x_i = r$, the agent is indifferent among all $\alpha^* \in [0,1]$ .

**Question 3**

The investor's problem is

$$\max_{\{\beta\}} 0.33\ln\left((30000-\beta)\times 1.05 + \beta\times 1.03\right) + 0.67\ln\left((30000-\beta)\times 1.05 + \beta\times 1.06\right)$$

where $\beta$ is the amount of wealth that the investor puts into the risky asset.

F.O.C.:

$$0.33\times\frac{-0.02}{31500-0.02\beta} + 0.67\times\frac{0.01}{31500+0.01\beta} = 0$$

Solving the above equation, we have $\beta^* = 15750$.

**Question 4**

Given the descriptions, it must be true that

$$p\ln(1,100,000) + (1-p)\ln(900,000) \geq \ln(1,000,000) \ ,$$

where $p$ is the probability that the person assigns to the Lakers winning.

Solving the inequality, we have

$$p \geq \frac{\ln(1,000,000) - \ln(900,000)}{\ln(1,100,000) - \ln(900,000)} = \frac{\ln\left(\frac{10}{9}\right)}{\ln\left(\frac{11}{9}\right)} \approx 0.525$$

Hence, the minimum probability he assigns to the Lakers winning the championship is 0.525.

**Question 5**
The individual's problem is to

$$\max_{\{x\}} \frac{1}{2}(w-x)^\rho + \frac{1}{2}(w+x\cdot s)^\rho$$

F.O.C.:

$$-\tfrac{1}{2}\rho(w-x)^{\rho-1} + \tfrac{\rho}{2}(w+x\cdot s)^{\rho-1}\cdot s = 0$$
$$\implies (w-x)^{\rho-1} = (w+x\cdot s)^{\rho-1}\cdot s$$
$$\implies w-x = (w+x\cdot s)\cdot s^{\frac{1}{\rho-1}}$$
$$\implies (w-x)s^{\frac{1}{1-\rho}} = (w+x\cdot s)$$
$$\implies w\cdot s^{\frac{1}{1-\rho}} - w = x\cdot s + x\cdot s^{\frac{1}{1-\rho}}$$
$$\implies x(s) = \frac{w\cdot s^{\frac{1}{1-\rho}} - w}{s+s^{\frac{1}{1-\rho}}}$$

Hence, she'll bet $x(s) = \frac{w\cdot s^{\frac{1}{1-\rho}} - w}{s+s^{\frac{1}{1-\rho}}}$.