Collegio Carlo Alberto

Economic Principles
Solutions to Problem Set 7

Question 1
Pareto efficient allocations $((x_1^1, x_2^1), (x_1^2, x_2^2))$ are characterized by the following conditions:

$$x_1^1 \geq x_2^1,\ x_1^2 \geq x_2^2,\ x_1^1 + x_1^2 = 20,\ x_2^1 + x_2^2 = 10.$$

Question 2
Since only relative prices matter, let's normalize $p_2 = 1$.
Consumer 1's problems is

$$\max x_1 x_2 \quad \text{s.t. } p_1 x_1 + x_2 = p_1 + 1$$

Hence, consumer 1's demands are

$$x_1^1 = \frac{p_1 + 1}{2p_1}$$
$$x_2^1 = \frac{p_1 + 1}{2}$$

Similarly, we can find consumer 2's demands:

$$x_1^2 = \frac{p_1 + 3}{2p_1}$$
$$x_2^2 = \frac{p_1 + 3}{2}$$

Market clearing conditions require that

$$x_1^1 + x_1^2 = e_1^1 + e_1^2 \Longrightarrow \frac{p_1 + 1}{2p_1} + \frac{p_1 + 3}{2p_1} = 2 \Longrightarrow p_1 = 2$$

Hence, the Walrasian equilibrium price is $\frac{p_1}{p_2} = 2$.Equilibrium allocation is $\left(\left(\frac{3}{4}, \frac{3}{2}\right), \left(\frac{5}{4}, \frac{5}{2}\right)\right)$.

Question 3
Let's normalize $p_1 = 1$.
Consumer 1's demands satisfy

$$x_1^1 = x_2^1 \text{ and } x_1^1 + p_2 x_2^1 = 1$$

Hence,

$$x_1^1 = x_2^1 = \frac{1}{1 + p_2}$$

Similarly,

$$\begin{aligned} x_2^2 &= x_3^2 \text{ and } p_2x_2^2 + p_3x_3^2 = p_2 \Longrightarrow x_2^2 = x_3^2 = \frac{p_2}{p_2+p_3} \\ x_1^3 &= x_3^3 \text{ and } x_1^3 + p_3x_3^3 = p_3 \Longrightarrow x_1^3 = x_3^3 = \frac{p_3}{1+p_3} \end{aligned}$$

Together with the market clearing conditions, we have

$$\begin{cases} \frac{1}{1+p_2} + \frac{p_3}{1+p_3} = 1 \\ \frac{1}{1+p_2} + \frac{p_2}{p_2+p_3} = 1 \end{cases}$$

This implies that $p_2 = p_3 = 1$.
Hence, the Walrasian equilibrium price is $(1,1,1)$ and allocation is $\left(\left(\frac{1}{2},\frac{1}{2},0\right),\left(0,\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},0,\frac{1}{2}\right)\right)$.

Question 4
Let $p_2 = 1$.
Consumer 1's problem is

$$\max \left[(x_1)^{-2} + \left(\frac{12}{37}\right)^3 (x_2)^{-2}\right]^{-1} \quad \text{s.t. } p_1x_1 + x_2 = p_1$$

We find

$$\begin{aligned} x_1^1 &= \frac{p_1}{p_1 + \frac{12}{37}(p_1)^{\frac{1}{3}}} \\ x_2^1 &= \frac{\frac{12}{37}(p_1)^{\frac{4}{3}}}{p_1 + \frac{12}{37}(p_1)^{\frac{1}{3}}} \end{aligned}$$

Consumer 2' problem is

$$\max \left[\left(\frac{12}{37}\right)^3 (x_1)^{-2} + (x_2)^{-2}\right]^{-1} \quad \text{s.t. } p_1x_1 + x_2 = 1$$

We find

$$\begin{aligned} x_1^2 &= \frac{1}{p_1 + \frac{37}{12}(p_1)^{\frac{1}{3}}} \\ x_2^2 &= \frac{\frac{37}{12}(p_1)^{\frac{1}{3}}}{p_1 + \frac{37}{12}(p_1)^{\frac{1}{3}}} \end{aligned}$$

Market clearing condition:

$$x_1^1 + x_1^2 = e_1^1 + e_1^2 \Longrightarrow \frac{p_1}{p_1 + \frac{12}{37}(p_1)^{\frac{1}{3}}} + \frac{1}{p_1 + \frac{37}{12}(p_1)^{\frac{1}{3}}} = 1 \Longrightarrow p_1 = 1 \text{ or } p_1 = \frac{27}{64} \text{or } p_1 = \frac{64}{27}$$

When $p_1 = 1, p_2 = 1$, equilibrium allocation is $\left(\left(\frac{37}{49},\frac{12}{49}\right),\left(\frac{12}{49},\frac{37}{49}\right)\right)$.
When $p_1 = \frac{27}{64}, p_2 = 1$, equilibrium allocation is $\left(\left(\frac{111}{175},\frac{27}{175}\right),\left(\frac{64}{175},\frac{148}{175}\right)\right)$.
When $p_1 = \frac{64}{27}, p_2 = 1$, equilibrium allocation is $\left(\left(\frac{148}{175},\frac{64}{175}\right),\left(\frac{27}{175},\frac{111}{175}\right)\right)$.