Collegio Carlo Alberto

Economic Principles
Solutions to Problem Set 8

Question 1
Let p denote the price of the coconuts and w denote the price of Robinson's time. Normalize $p = 1$.
Given the production function, the producer will choose l to maximize $p\sqrt{l} - wl$. Hence, the producer's labor demand

$$l^p = \left(\frac{1}{2w}\right)^2$$

and coconut supply

$$y^p = \frac{1}{2w}$$

The producer 's profit is

$$\frac{1}{2w} - \left(\frac{1}{2w}\right)^2 w = \frac{1}{4w}$$

The consumer's problem is

$$\max x^{\frac{1}{3}} h^{\frac{2}{3}} \quad \text{s.t. } px + wh = 24w + \frac{1}{4w}$$

Hence, the consumer's coconut demand

$$x^c = \frac{24w + \frac{1}{4w}}{3}$$

and leisure demand

$$h^c = \frac{48w + \frac{1}{2w}}{3w}$$

Hence, the consumer's labor supply is

$$24 - h^c = 24 - \frac{48w + \frac{1}{2w}}{3w}$$

For the coconut market to clear,

$$y^p = x^c \Longrightarrow \frac{1}{2w} = \frac{24w + \frac{1}{4w}}{3} \Longrightarrow w = \frac{\sqrt{30}}{24}$$

Hence, the Walrasian equilibrium prices are

$$w = \frac{\sqrt{30}}{24} \text{ and } p = 1$$

In the coconut market, $y^p = x^c = \frac{2}{5}\sqrt{30} = 2.1909$
Also, labor supply = labor demand = $24 - \frac{96}{5} = 4.8$.
The consumer's leisure consumption = $\frac{96}{5} = 19.2$

Question 2

Robinson's problem is

$$\max x^{\frac{1}{3}} h^{\frac{2}{3}} \quad \text{s.t. } x = \sqrt{24 - h}$$

We have F.O.C. (where we first transformed the utility function in logarithmic form):

$$-\frac{1}{6(24-h)} + \frac{2}{3h} = 0 \Longrightarrow h^* = \frac{96}{5} \text{ and } x^* = \sqrt{24 - \frac{96}{5}} = \frac{2}{5}\sqrt{30}$$

Hence, we get the same answer as in question 1.

Question 3

Denote p_1, p_2, w as the prices for good 1, good 2 and labor, respectively.
Normalize $w = 1$.
Firm 1's problem is

$$\max_{\{l_1\}} p_1 \sqrt{l_1} - w l_1$$

F.O.C.

$$\frac{p_1}{2\sqrt{l_1}} - w = 0$$

Hence, firm 1's labor demand is $\frac{(p_1)^2}{4}$ and the supply of y_1 is $\frac{p_1}{2}$.
Firm 1's profit

$$\pi_1 = \frac{p_1}{2} \cdot p_1 - \frac{(p_1)^2}{4} \times 1 = \frac{(p_1)^2}{4}$$

Similarly, firm 2's problem is to choose l_2 to maximize $(p_2 - w)l_2$.
When $p_2 > 1$, firm 2's demand for labor is ∞ and this is impossible in an equilibrium.
When $p_2 < 1$, supply of $y_2 = 0$.
When $p_2 = 1$, supply of y_2 can be anything within the resource constraint.
Hence, firm 2's profit $\pi_2 = 0$.
Consumer 1's problem is

$$\max x_1 x_2 \text{ s.t. } p_1 x_1 + p_2 x_2 = 6 + \frac{(p_1)^2}{8}$$

Hence, consumer 1's demands are

$$\begin{aligned} x_1^1 &= \frac{6 + \frac{(p_1)^2}{8}}{2p_1} \\ x_2^1 &= \frac{6 + \frac{(p_1)^2}{8}}{2p_2} \end{aligned}$$

Similarly, we find that consumer 2's demands are

$$\begin{aligned} x_1^2 &= \frac{6 + \frac{(p_1)^2}{8}}{2p_1} \\ x_2^2 &= \frac{6 + \frac{(p_1)^2}{8}}{2p_2} \end{aligned}$$

Since the demand for good 2 is positive, it has to be the case that $p_2 = 1$ for the market of good 2 to clear.
For the market of good 1 to clear,

$$x_1^1 + x_1^2 = \frac{p_1}{2} \Longrightarrow \frac{6 + \frac{(p_1)^2}{8}}{p_1} = \frac{p_1}{2} \Longrightarrow p_1 = 4$$

In summary, the Walrasian equilibrium prices are $p_1 = 4, p_2 = 1, w = 1$.
Firm 1's labor demand $l_1 = 4$, supply of good 1 $y_1 = 2$.
Firm 2's labor demand $l_2 = 8$, supply of good 2 $y_2 = 8$.
Each consumer supplies 6 units of labor and each consumes 1 unit of good 1 and 4 units of good 2.

Question 4
(a) Given the assumptions, for a utility maximizing consumer, $MRS_{ij} = \frac{p_i}{p_j}$ in equilibrium. Since $\frac{p_i}{p_j}$ is the same for each consumer, for any two consumption goods i and j, the MRS between any two goods is the same for each consumer.
(b) Given the assumptions, a profit maximizing firm that uses inputs k and h in the production will use them such that $MRTS_{kh} = \frac{p_k}{p_h}$. Since the input prices are the same for each firm, the $MRTS$ between any two inputs is the same for every firm using those inputs.

Question 5
Let $p_2 = 1$.
For consumer 1,

$$\max x_1^{\alpha} x_2^{1-\alpha} \quad \text{s.t. } p_1 x_1 + x_2 = p_1 e_1^1 + e_2^1$$

Hence, the consumer 1's demands are

$$\begin{aligned} x_1^1 &= \frac{\alpha \left(p_1 e_1^1 + e_2^1\right)}{p_1} \\ x_2^1 &= \frac{(1-\alpha)\left(p_1 e_1^1 + e_2^1\right)}{1} \end{aligned}$$

Similarly, we can find that consumer 2's demands are

$$\begin{aligned} x_1^2 &= \frac{\alpha (p_1 e_1^1 + e_2^1)}{p_1} \\ x_2^2 &= \frac{(1-\alpha)\left(p_1 e_1^2 + e_2^2\right)}{1} \end{aligned}$$

Market clearing condition requires that

$$x_1^1 + x_1^2 = \frac{\alpha\left(p_1\left(e_1^1 + e_1^2\right) + e_2^1 + e_2^2\right)}{p_1} = \frac{\alpha\left(10p_1 + 10\right)}{p_1} = 10 \Longrightarrow p_1 = \frac{\alpha}{1-\alpha}$$

Hence, the Walrasian equilibrium prices are $p_1 = \frac{\alpha}{1-\alpha}, p_2 = 1$.
Now, we want to find e^1 and e^2 to support allocation $\left((5,5),(5,5)\right)$.
Fix e_1^1. Then,

$$x_1^1 = 5 \Longrightarrow \frac{\alpha\left(p_1 e_1^1 + e_2^1\right)}{\frac{\alpha}{1-\alpha}} = 5 \Longrightarrow e_2^1 = \frac{5 - \alpha e_1^1}{1-\alpha}$$

Also, $e_1^2 = 10 - e_1^1$, $e_2^2 = 10 - \frac{5-\alpha e_1^1}{1-\alpha} = \frac{5-10\alpha+\alpha e_1^1}{1-\alpha}$.
We require that $0 \le e_1^1 \le 10$ and $0 \le e_2^1 = \frac{5-\alpha e_1^1}{1-\alpha} \le 10$.

$0 \le \frac{5-\alpha e_1^1}{1-\alpha} \Longrightarrow e_1^1 \le \frac{5}{\alpha}$ so a sufficient condition for $[e_1^1 \le 10 \; \& e_1^1 \le \frac{5}{\alpha}]$ is e_1^1 less than the minimum of the two.
$e_2^1 = \frac{5-\alpha e_1^1}{1-\alpha} \le 10 \Longrightarrow e_1^1 \ge 10 - \frac{5}{\alpha}$ and a sufficient condition for $[e_1^1 \ge 0 \; \& \; e_1^1 \ge 10 - \frac{5}{\alpha}]$ is e_1^1 greater than the maximum of the two.
Hence, $\max(0, 10 - \frac{5}{\alpha}) \le e_1^1 \le \min\left(10, \frac{5}{\alpha}\right)$.
Also, since $e^1 \ne e^2$, we require that $e_1^1 \ne 5$.