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Economic Principles

Solutions to Problem Set 9

Question 1

(a) The Pareto-efficient allocations solve to the following problem (where we took the square root of u^A and take the logarithm, both monotonic transformations):

$$\begin{gathered}\max \ln x_1^A + \ln x_2^A \\ \text{s.t.} \quad \ln x_1^B + \ln x_2^B \geq u \\ x_1^A + x_1^B = 18 \\ x_2^A + x_2^B = 14 \\ x_1^A \geq 0, x_2^A \geq 0,\ x_1^B \geq 0,\ x_2^B \geq 0\end{gathered}$$

We can solve the problem by ignoring the non-negativity constraints, and once we find the uncostrained solution, check that it satisfies the constraints we ignored.. Moreover we can solve for (x_1^B, x_2^B) from the feasibility constraints and our problem becomes a max problem in the two variables (x_1^A, x_2^A) with the constraint that $(x_1^A, x_2^A) \in [0, 18] \times [0, 14]$.Therefore we form the Lagrangean:

$$L = \ln x_1^A + \ln x_2^A + \lambda[-u + \ln(18 - x_1^A) + 2\ln(14 - x_2^A)]$$

The first order conditions are:

$$\frac{1}{x_1^A} = \lambda \frac{1}{18 - x_1^A}$$

$$\frac{1}{x_2^A} = \lambda \frac{2}{14 - x_2^A}$$

Diving the first equation by the second we obtain:

$$\frac{x_2^A}{x_1^A} = \frac{1}{2} \frac{14 - x_2^A}{18 - x_1^A}$$

Solving for x_2^A as a function of x_1^A we obtain:

$$x_2^A = \frac{14 x_1^A}{36 - x_1^A}$$

Notice that the solution satisfies the constraint $x_2^A \in [0, 14]$ when $x_1^A \in [0, 18]$, therefore the non-negativity constraint is never binding. Notice that this was expected since for Cobb-Douglas utility the indifference curves never touch the axis. therefore a point of tangency can never happen outiside the Edgeworth box. This is our set of Pareto efficient allocations.
(b1) To characterize the core, we simply note that when there are only 2 agents, the core is the set of PE allocations that guarantee each agent at least as much utility as she would have derived from her endowment. Since both utility functions are monotonic transformations of Cobb-Douglas utility, we can work with utilities $u^A(x_1^A, x_2^A) = x_1^A x_2^A$ and $u^B(x_1^B, x_2^B) = x_1^B (x_2^B)^2$.
Since $u^A(e_1^A, e_2^A) = u^A(15, 6) = 90$ and $u^B(e_1^B, e_2^B) = u^B(3, 8) = 3 * 8^2 = 192$, the core can be characterized by $x_2^A = \frac{14x_1^A}{36-x_1^A}$ along with the two conditions

$$u^A(x_1^A, x_2^A) \geq 90$$

$$u^B(x_1^B, x_2^B) \geq 192$$

(b2)
In point b1 we found that the core is characterized by:

$$\begin{aligned} u^A(x_1^A, \frac{14x_1^A}{36 - x_1^A}) &\geq 90 \\ u^B(18 - x_1^A, 14 - \frac{14x_1^A}{36 - x_1^A}) &\geq 192 \end{aligned}$$

The system we need to solve is therefore:

$$\begin{cases} x_1^A \frac{14x_1^A}{36-x_1^A} - 90 \geq 0 \\ (18 - x_1^A)(14 - \frac{14x_1^A}{36-x_1^A})^2 - 192 \geq 0 \end{cases}$$

Using a computer, we find that the solution is the interval $I = [12.33, 12.93]$. Therefore the core allocations are characterized by the following one parameter system:

$$\begin{gathered} (x, \tfrac{14x}{36-x}) \text{ for consumer } A \\ (18 - x, 14 - \tfrac{14x}{36-x}) \text{ for consumer } B \\ x \in I \end{gathered}$$

(c) We now find the Walrasian equilibrium prices and allocation. Normalize $p_1 = 1$ and let p represent p_2.
Both agents have Cobb-Douglas utility functions. Their incomes are $y^A = 15 + 6p$ and $y^B = 3 + 8p$. Therefore their demand functions are:

$\big(x_1^A(p), x_2^A(p)\big) = (\frac{1}{2}y^A, \frac{1}{2}\frac{y^A}{p}) = \Big(\frac{1}{2}(15+6p), \frac{1}{2p}(15+6p)\Big)$
$(x_1^B(p), x_2^B(p)) = (\frac{1}{3}y^B, \frac{2}{3p}y^B) = \Big(\frac{1}{3}(3+8p), \frac{2}{3p}(3+8p)\Big)$
In equilibrium we require that $x_1^A(p)+x_1^B(p) = e_1^A+e_1^B = 18$, so this yields $p = \frac{57}{34}$, and so the equilibrium allocation is $x^A = (\frac{213}{17}, \frac{142}{19})$ and $x^B = (\frac{93}{17}, \frac{124}{19})$. Plugging these into the utility functions gives us $u^A(\frac{213}{17}, \frac{142}{19}) = \frac{213}{17}\frac{142}{19} = 93.641$and $u^B(\frac{93}{17}, \frac{124}{19}) = \frac{93}{17}(\frac{124}{19})^2 = 233.01$ Since every Walrasian equilibrium is PE and in this particular equilibrium each agent derives at least as much utility as she would have derived from consuming her endowment, we can say this allocation belongs to the core.

Question 2
To find the Walrasian equilibrium prices and allocation, we need to derive the Marshallian demands of the two agents. One way to do this is to first derive the Hicksian demands from the expenditure function using Shephard's Lemma and then to convert these into Marshallian demands.
Thus, we know that $h_1^A(p,u) = \frac{\partial e^A}{\partial p_1}$ and $h_2^A(p,u) = \frac{\partial e^A}{\partial p_2}$. So:

$$h_1^A(p,u) = \frac{1}{3}[3(1.5)^2 p_1^2 p_2 e^u]^{-2/3}[2.3(1.5)^2 p_1 p_2 e^u] = \frac{2}{3p_1}e^A(p,u).$$

Similarly,

$$h_2^A(p,u) = \tfrac{1}{3p_2}e^A(p,u),$$
$$h_1^B(p,u) = \tfrac{1}{3p_1}e^B(p,u),$$
$$h_2^B(p,u) = \tfrac{2}{3p_2}e^B(p,u).$$

Recall that $h(p,u) = x(p, e(p,u))$. So $x_1^A(p, e^A(p,u)) = \frac{2}{3p_1}e^A(p,u)$. Substituting y^A for $e(p,u)$, we have $x_1^A(p,y^A) = \frac{2}{3p_1}y^A$. Similarly, we derive $x_2^A(p,y^A) = \frac{1}{3p_2}y^A$, $x_1^B(p,y^B) = \frac{1}{3p_1}y^B$ and $x_2^B(p,y^B) = \frac{2}{3p_2}y^B$. We're almost done: normalize $p_1 = 1$ and let p represent p_2 and now note that $y^A = 10$ and $y^B = 10p$. In equilibrium we must have $x_1^A(p,y^A) + x_1^B(p,y^B) = 10$, which means $\frac{2}{3}y^A + \frac{1}{3}y^B = 10$. Solving this equation yields $p = 1$. The corresponding equilibrium allocations are $x^A = (20/3, 10/3)$ and $x^B = (10/3, 20/3)$.

Question 3
(a)
The Edgeworth box is a square with sides of lenght 10.
(b) To derive the core allocations, we again find the PE allocations that give each agent at least as much utility as her endowment (again note that this characterization is only valid when there are 2 agents). The PE allocations are characterized by:

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} \tag{1}$$

$$x_1^A + x_1^B = 10 \tag{2}$$

$$x_2^A + x_2^B = 10 \tag{3}$$

In exercise 1 we made the observation that when utility is Cobb-Douglas for both consumer, the point of tangency can never occur outside the Edgeworth box, therefore the tangency condition completely characterizes the set of PE allocations.
In the Edgeworth box, the set of PE allocations is therefore the diagonal line connecting the two origins, and so every allocation on this line satisfies $x_1^A = x_2^A$ and $x_1^B = x_2^B = 10 - x_1^A = 10 - x_2^A$. We also have $u^A(e_1^A, e_2^A) = u^A(8,2) = 16$ and $u^B(e_1^B, e_2^B) = u^A(2,8) = 16$. The PE allocation that gives consumer A this level of utility gives her (4,4), while the same is true for consumer B. Thus the core is the set of points on the diagonal line that lie between (and include) (4,4) and (6,6) (as measured from the origin of consumer A). This is of course just the contract curve.
(c) The allocation ((4,4),(6,6)) lies on the diagonal in the Edgeworth box and is therefore PE. Also, in this allocation, consumer A obtains utility 16 and consumer B obtains utility 36, and since both of them are at least as well off as if they had simply consumed their own endowments, this allocation must lie in the core.
(d) I show that the proposed allocation does not belong to the core, by finding a blocking coalition. Call the two consumers of type 1, A and B, and call the two consumers of type 2, C and D. Consider the coalition formed the consumers A, B and C and the following allocation for the members of this coalition ((5,4), (4,4) (9,4)). First notice that this is a feasible allocation for this coalition since $x_1^A + x_1^B + x_1^C = e_1^A + e_1^B + e_1^C$ and $x_2^A + x_2^B + x_2^C = e_2^A + e_2^B + e_2^C$. Now note that this allocation makes consumer A strictly better off than in the original allocation, while consumers B and C are exactly as well off as before. Thus this particular coalition can block the proposed allocation and so it cannot belong to the core.