You have 90 minutes to complete this exam. Please answer the following three questions. Be sure to allocate your time in proportion to the points. Always justify your answers by providing a formal proof or a detailed argument. Good luck.

1. [30 points] Consider the following Tullock contest in which \( n \) players compete for a prize \( v > 0 \). Each player \( i = 1, \ldots, n \) chooses an effort level \( x_i \geq 0 \) (the players move simultaneously). The prize is awarded to player \( i \) with probability

\[
\frac{x_i^\alpha}{\sum_{j=1}^n x_j^\alpha}
\]

if \( \sum_{j=1}^n x_j^\alpha > 0 \). The parameter \( \alpha \) is strictly positive. If all the players choose an effort level equal to zero, then they are all equally likely to win the prize.

The payoff of a player who exerts an effort level equal to \( x \) is equal to \( v - x \) if he wins the prize, and equal to \( -x \) if he does not win the prize.

Find the symmetric Nash equilibrium (in pure strategies).

1. [30 points] A seller has to invest \( c(q) = q^2 \) to produce a good of quality \( q \). There are two buyers and their valuation of a good of quality \( q \) is equal to \( v(q) = q \).

The timing of the game is as follows. First, the seller chooses the quality of the good \( q \) and makes the investment \( c(q) \) (the seller can produce only one good). Then the buyers observe \( q \) and make simultaneous offers. The offer of buyer \( i = 1, 2 \), is the price \( p_i \geq 0 \) that he is willing to pay. Finally, the seller decides which offer to accept.

The seller’s payoff is equal to the difference between his revenues and the investment cost. If a buyer purchases a good of quality \( q \) at the price \( p \), then he obtains a payoff equal to \( v(q) - p \). The payoff of a buyer who does not purchase the good is equal to zero.

Find a subgame perfect equilibrium of the game.

2. [40 points] Two players compete for a prize by choosing (simultaneously) non-negative effort levels. The prize has a value equal to one and is awarded to the player who exerts the largest effort level. The two players are equally likely to get the prize if they choose the same effort level.
The cost of the effort level is private information. In particular, each player $i$ has a type $t_i$ which is distributed over the interval $[0, \sqrt{6}]$ with a density function $f(t_i) = \frac{1}{2} t_i^2$. The players’ types are independent.

Consider player $i = 1, 2$, with type $t_i$ and assume that his effort level is $x_i \geq 0$. The player’s payoff is equal to $1 - t_i x_i$ if he wins the prize, and equal to $-t_i x_i$ if he does not win the prize.

Find a symmetric Bayesian Nash equilibrium (in pure strategies). (Assume that the equilibrium strategy is differentiable and strictly decreasing. What is the effort level of type $t_i = \sqrt{6}$?)