1. A seller owns an object that a buyer wants to buy. The value of the object to the seller is $c$. The value of the object to the buyer is private information. The buyer’s valuation $v$ is a random variable distributed over the interval $[0, V]$ according to the (continuous) c.d.f. $F$. Assume that $[1 - F(v)]/f(v)$ is a decreasing function of $v$. The von Neumann-Morgenstern utility of a type $v$ from getting a unit at price $p$ is $v - p$ and the utility of no purchase in 0.

(i) Suppose the seller is constrained to charge just one price. Show that the profit maximizing price satisfies $p = c + [1 - F(p)]/f(p)$.

(ii) Suppose that the seller can commit to a menu of offers $[q(v), p(v)]$, where $q(v)$ is the probability with which a consumer who chooses offer $v$ will get a unit, and $p(v)$ is the consumer’s (expected) payment. Prove that the menu that maximizes the seller’s profit consists of a single price, which is the one found in (i), and that any buyer can get the good at this price with probability 1.

2. Consider the following auction environment. A seller has a single object for sale and can commit to any selling mechanism (the seller’s valuation of the object is zero). There are two potential bidders, indexed by $i = 1, 2$. The valuation of the object of bidder $i = 1, 2$ is denoted by $v_i$ and is distributed uniformly over the unit interval. Valuations are independent between the two bidders. Bidder 1 knows her own valuation $v_1$. However, bidder 2 does not know $v_2$.

The bidders’ payoffs are as follows. Suppose bidder $i = 1, 2$ has type $v_i$ and pays the amount $t_i$ to the seller. Her payoff is equal to $v_i - t_i$ if she gets the object, and equal to $-t_i$ otherwise.

(i) Construct the optimal direct mechanism for the seller (i.e., find the incentive compatible, individually rational mechanism that maximizes the seller’s expected revenues). Compute the seller’s revenues.

(ii) Can you find a simple indirect mechanism that gives to the seller the same expected revenues as the optimal direct mechanism?

3. A seller has a unit for sale. Its quality is either high ($H$) or low ($L$). The quality is known to the seller but not to the buyer, whose prior probability that the quality is
high is 1/2. Their valuations of the unit are as follows.

<table>
<thead>
<tr>
<th></th>
<th>Quality H</th>
<th>Quality L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buyer</td>
<td>V</td>
<td>2</td>
</tr>
<tr>
<td>Seller</td>
<td>7</td>
<td>0</td>
</tr>
</tbody>
</table>

where $V > 7$. Thus, the utility to the buyer of getting the unit at price $p$ is $2 - p$ if it is of the low quality, and $V - p$ if it is of the high quality. Similarly, the utility to the seller is $p$ and $p - 7$, respectively.

(i) Find the ex-post efficient outcomes.

(ii) Identify the range of $V$ (above 7) for which there is, and the range of $V$ for which there is no incentive compatible, individual rational mechanism that will achieve the ex-post efficient outcome.

(iii) Describe the best outcome (in the maximizing of the sum of expected utilities) that can be achieved for each $V$ (above 7) and the mechanism that achieves it.

HINT: A mechanism for this Bayesian bargaining problem consists of a pair of functions $q : \{L, H\} \rightarrow [0, 1]$ and $t : \{L, H\} \rightarrow \mathbb{R}$, where $q(i)$ is the probability that the object will be sold to the buyer and $t(i)$ is the expected net payment from the buyer to the seller if $i = L, H$ is the type reported by the seller to a mediator.

4. A seller owns an object that a buyer wants to buy. The quality of the object is a random variable $v$, with support $[0, 1]$ and distribution function $F(v) = v^{\alpha}$, where $\alpha > 0$. The seller knows the quality of the object but the buyer does not. When the quality of the object is $v$, the value of the object is $v$ to the seller and $zv$ to the buyer, where $z > 1$. Thus, if the object of quality $v$ is traded at price $p$, the seller gets $p - v$ and the buyer gets $zv - p$. Both players have utility equal to zero if there is no trade.

Consider the function $G : (0, \infty) \times (1, \infty) \rightarrow [0, 1]$ defined as follows. For each pair $(\alpha, z)$ construct the incentive-compatible individually rational mechanism that maximizes the (ex-ante) probability of trade. Denote this probability by $G(\alpha, z)$. Derive the function $G$.

(N.B. If the probability of trade is $q(v)$ when the quality is $v$, then the (ex-ante) probability of trade is equal to $\int_{0}^{1} q(v) dF(v)$.)

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