Collegio Carlo Alberto

Game Theory
Problem Set 4

1. (Hotelling's model) Consumers are uniformly distributed along a boardwalk that is 1 mile long. Ice-cream prices are regulated, so consumers go to the nearest vendor because they dislike walking. Suppose that at the regulated prices all consumers will purchase an ice cream even if they have to walk a full mile. The vendors choose their locations simultaneously. If more than one vendor are at the same location, they split the business evenly.

(a) Consider the game with two ice-cream vendors. Find all pure strategy Nash equilibria.

Hint: The set of actions available to player $i = 1, 2$ is $S_i = [0, 1]$ and the payoff function is given by:

$$u_i(s_i, s_j) = \begin{cases} \frac{s_i+s_j}{2} & \text{if } s_i < s_j \\ \frac{1}{2} & \text{if } s_i = s_j \\ 1 - \frac{s_i+s_j}{2} & \text{if } s_i > s_j \end{cases}.$$

(b) Show that with three vendor, no pure strategy Nash equilibrium exists.

2. (Air Strike) Army A has a single plane with which it can strike one of three possible targets. Army B has one anti-aircraft gun that can be assigned to one of the targets. The value of the target k is v_k, with $v_1 > v_2 > v_3$. Army A can destroy a target only if the target is undefended and A attacks it. Army A wishes to maximize the expected value of the damage and Army B wishes to minimize it. Formulate the situation as a strategic form game. Does the game have pure strategy Nash equilibria? Find the mixed strategy Nash equilibria. What happens to the equilibrium strategies as $v_3 \to 0$ (i.e., v_3 becomes arbitrarily small) and v_1 and v_2 remain constant? Present your intuition.

3. (First price auction with different valuations) Consider the following auction. There are $n > 1$ players who compete for an object. Player i's valuation of the object is v_i, and $v_1 > v_2 > \ldots v_n > 0$. The players simultaneously submit their bids. The player with the highest bid gets the prize and pays her bid (if two or more players submit the highest bid, the player with the lowest index gets the object). A player does not pay anything if she does not get the prize. Show that in all Nash equilibria player 1 obtains the object.

4. (A simple Bayesian game) Consider a Bayesian game in which player 1 may be either type a or type b, where type a has probability .9 and type b has probability .1. Player 2 has no private information. Depending on player 1's types, the payoffs to the two players depend on their actions in $A_1 = \{U, D\}$ and $A_2 = \{L, R\}$ as shown in the following table.

$t_1 = a$

	L	R
U	$2, 2$	$-2, 0$
D	$0, -2$	$0, 0$

$t_1 = b$

	L	R
U	$0, 2$	$1, 0$
D	$1, -2$	$2, 0$

Compute all Bayesian Nash equilibria of this game.

5. (An exchange game) Each of the two players receives a ticket on which there is a number in the finite set $\{x_1, x_2, \ldots, x_m\}$, where $x_1 < x_2 < \ldots < x_m$. The number on a player's ticket is the size of a prize that she may receive. The two prizes are identically and independently distributed, with distribution function F. Each player knows only the number on her own ticket. Each player is asked independently and simultaneously whether she wants to exchange her prize for the other player's prize. If both players agree then the prizes are exchanged; otherwise each player receives her own prize. The players' objective is to maximize their expected payoffs. Model this situation as a Bayesian game and show that in any Bayesian Nash equilibrium the highest prize that either player is willing to exchange is the smallest possible prize x_1.