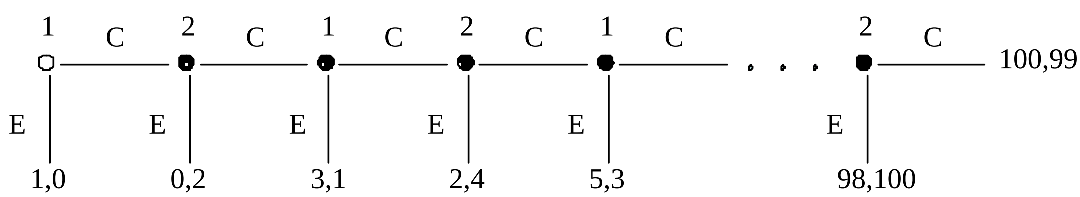
Collegio Carlo Alberto

Game Theory

Problem Set 7

1. (The centipede game) In class we showed that in the unique subgame perfect equilibrium of the centipede game (see the figure below) both players exit in every information set.



(a) Does the game admit Nash equilibria which are not subgame perfect?

(b) Is the subgame-perfect equilibrium outcome the unique Nash equilibrium outcome of the centipede game?

2. (Sequential bargaining) Suppose the players in Rubinstein's infinite-horizon bargaining game have different discount factors: δ_1 for player 1 and δ_2 for player 2. Adapt the argument illustrated in class to show that in the unique subgame-perfect equilibrium outcome player 1 offers the settlement $\left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ to player 2 who accepts.

3. (Gibbons, Exercise 2.1, page 130) Suppose a parent and child play the following game, first analyzed by Becker. First, the child takes an action, A, that produces income for the child, $I_C(A)$, and income for the parent, $I_P(A)$. (Think of $I_C(A)$ as the child's income net of any cost of the action A.) Second, the parent observes the incomes I_C and I_P and then chooses a bequest, B, to leave to the child. The child's payoff is $U(I_C + B)$; the parent's is $V(I_P - B) + kU(I_C + B)$, where $k > 0$ reflects the parent's concern for the child's well-being. Assume that: the action is a nonnegative number, $A \geqslant 0$; the income functions $I_C(A)$ and $I_P(A)$ are strictly concave and maximized at $A_C > 0$ and $A_P > 0$, respectively; the bequest B can be positive or negative; and the utility functions U and V are increasing and strictly concave. Prove the "Rotten Kid"

Theorem: in the subgame-perfect equilibrium outcome, the child chooses the action that maximizes the family's aggregate income, $I_C(A) + I_P(A)$, even though only the parent's payoff exhibits altruism.

4. (Gibbons, Exercise 2.4, page 131) Two partners would like to complete a project. Each partner receives the payoff V when the project is completed but neither receives any payoff before completion. The cost remaining before the project can be completed is R. Neither partner can commit to making a future contribution towards completing the project, so they decide to play the following two-period game: In period one partner 1 chooses to contribute c_1 towards completion. If this contribution is sufficient to complete the project then the game ends and each partner receives V. If this contribution is not sufficient to complete the project (i.e., $c_1 < R$) then in period two partner 2 chooses to contribute c_2 towards completion. If the (undiscounted) sum of the two contributions is sufficient to complete the project then the game ends and each partner receives V. If this sum is not sufficient to complete the project then the game ends and both partners receive zero.

Each partner must generate the funds for a contribution by taking money away from other profitable activities. The optimal way to do this is to take money away from the least profitable alternative first. The resulting (opportunity) cost of a contribution is thus convex in the size of the contribution. Suppose that the cost of contribution c is c^2 for each partner. Assume that partner 1 discounts second-period benefits by the discount factor $\delta \in (0,1)$. Compute the unique backwards-induction outcome of this two-period contribution game for each triple of parameters $\{V, R, \delta\}$.

5. (Gibbons, Exercise 2.5, page 132) Suppose a firm wants a worker to invest in a firm-specific skill, S, but the skill is too nebulous for a court to verify whether the worker has acquired it. (For example, the firm might ask the worker to "familiarize yourself with how we do things around here," or "become an expert on this new market we might enter.") The firm therefore cannot contract to repay the worker's cost of investing: even if the worker invests, the firm can claim that the worker did not invest, and the court cannot tell whose claim is true. Likewise, the worker cannot contract to invest if paid in advance.

It may be that the firm can use the (credible) promise of a promotion as an incentive for the worker to invest, as follows. Suppose that there are two jobs in the firm, one easy (E) and the other difficult (D), and that the skill is valuable on both jobs but more so on the difficult job: $y_{D0} < y_{E0} < y_{ES} < y_{DS}$, where y_{ij} is the worker's output in job i ($= E$ or D) when the worker's skill level is j ($= 0$ or S). Assume that the firm can commit to paying different wages in the two jobs, w_E and w_D, but that neither wage can be less than the worker's alternative wage, which we normalize to zero.

The timing of the game is as follows: At date 0 the firm chooses w_E and w_D and the worker observes these wages. At date 1 the worker joins the firm and can acquire the skill S at cost C. (We ignore production and wages during the first period. Since the worker has not yet acquired the skill, the efficient assignment is to job E.) Assume that $y_{DS} - y_{E0} > C$, so that it is efficient for the worker to invest. At date 2 the firm observes whether the worker has acquired the skill and then decides whether to promote the worker to job D for the worker's second (and last) period of employment.

The firm's second-period profit is $y_{ij} - w_i$ when the worker is in job i and has skill level j. The worker's payoff from being in job i in the second period is w_i or $w_i - C$, depending on whether the worker invested in the first period. Solve for the subgame-perfect equilibrium outcome.