1. [30 points] Consider the following Tullock contest in which \( n \) players compete for a prize \( v > 0 \). Each player \( i = 1, \ldots, n \) chooses an effort level \( x_i \geq 0 \) (the players move simultaneously). The prize is awarded to player \( i \) with probability

\[
\frac{x_i^\alpha}{\sum_{j=1}^{n} x_j^\alpha}
\]

if \( \sum_{j=1}^{n} x_j^\alpha > 0 \). The parameter \( \alpha \) is strictly positive. If all the players choose an effort level equal to zero, then they are all equally likely to win the prize.

The payoff of a player who exerts an effort level equal to \( x \) is equal to \( v - x \) if he wins the prize, and equal to \(-x\) if he does not win the prize.

Find the symmetric Nash equilibrium (in pure strategies).

The strategy profile \((0, \ldots, 0)\) is not a Nash equilibrium. In fact, with this strategy profile each player obtains a payoff equal to \( v/n \). However, if a player deviates and chooses a positive level of effort \( \varepsilon \), his payoff is equal to \( v - \varepsilon \). For \( \varepsilon < v^{n-1}/n \) the deviation is strictly profitable. Therefore, we look for a symmetric Nash equilibrium in which each player exerts a positive level of effort \( \bar{x} \). Then, we must have:

\[
\bar{x} = \arg \max_{x \geq 0} \frac{v x^\alpha}{x^\alpha + (n - 1) \bar{x}^\alpha} - x
\]

From the first order conditions we obtain:

\[
v \frac{\alpha n \bar{x}^{\alpha-1} \bar{x}^\alpha - \alpha \bar{x}^{\alpha-1} \bar{x}^{\alpha}}{n^2 \bar{x}^{2\alpha}} = 1
\]

Therefore, we have

\[
\bar{x} = \frac{v \alpha (n - 1)}{n^2}
\]

2. [30 points] A seller has to invest \( c(q) = q^2 \) to produce a good of quality \( q \). There are two buyers and their valuation of a good of quality \( q \) is equal to \( v(q) = q \).

The timing of the game is as follows. First, the seller chooses the quality of the good \( q \) and makes the investment \( c(q) \) (the seller can produce only one good). Then the
buyers observe \( q \) and make simultaneous offers. The offer of buyer \( i = 1, 2 \), is the price \( p_i \geq 0 \) that he is willing to pay. Finally, the seller decides which offer to accept.

The seller’s payoff is equal to the difference between his revenues and the investment cost. If a buyer purchases a good of quality \( q \) at the price \( p \), then he obtains a payoff equal to \( v(q) - p \). The payoff of a buyer who does not purchase the good is equal to zero.

Find a subgame perfect equilibrium of the game.

Suppose that the seller has produced a good of quality \( q \). In any subgame perfect equilibrium, every buyer \( i = 1, 2 \), makes the offer \( p_i = q \). For simplicity, we restrict attention to equilibria in which the buyers play pure strategies (the argument can be easily extended if we allow the buyers to mix). Clearly, if \( p_i > p_j \), then the seller will accept \( p_i \). This implies that the two offers must coincide (if \( p_i > p_j \), then buyer \( i \) has an incentive to lower his offer). However, we cannot have \( p_1 = p_2 = p < q \). In fact, at least one buyer obtains a payoff strictly smaller than \( q - p > 0 \). The buyer can obtain a payoff arbitrarily close to \( q - p \) by offering \( p + \varepsilon \) (with \( \varepsilon \) positive and close to zero). Also, we cannot have \( p_1 = p_2 = p > q \). In fact, at least one buyer obtains a negative payoff. However, notice that each buyer can obtain a weakly positive payoff by offering zero.

We now turn to the initial period. The seller knows that if he produces a good of quality \( q \), then his revenues will be equal to \( q \). Thus, he chooses \( q \) to maximize \( q - q^2 \). The solution is \( q^* = \frac{1}{2} \).

To sum up, all subgame perfect equilibria share the following properties. In the first period, the seller produces \( q^* = \frac{1}{2} \). If the quality is \( q \), both buyers offer \( q \). The seller accepts the largest offer (he can mix in any way when the two offers coincide).

3. [40 points] Two players compete for a prize by choosing (simultaneously) non-negative effort levels. The prize has a value equal to one and is awarded to the player who exerts the largest effort level. The two players are equally likely to get the prize if they choose the same effort level.

The cost of the effort level is private information. In particular, each player \( i \) has a type \( t_i \) which is distributed over the interval \([0, \sqrt{6}]\) with a density function \( f(t_i) = \frac{1}{2}t_i^2 \). The players’ types are independent.

Consider player \( i = 1, 2 \), with type \( t_i \) and assume that his effort level is \( x_i \geq 0 \). The player’s payoff is equal to \( 1 - t_i x_i \) if he wins the prize, and equal to \(-t_i x_i \) if he does not win the prize.

Find a symmetric Bayesian Nash equilibrium (in pure strategies). (Assume that the equilibrium strategy is differentiable and strictly decreasing. What is the effort level of type \( t_i = \sqrt{6} \)?)
Let \( g : [0, \sqrt{6}] \rightarrow \mathbb{R}_+ \) denote the players’ strategy. For every \( t \in [0, \sqrt{6}] \), the function \( g \) must satisfy:

\[
t = \arg \max_{y \in [0, \sqrt{6}]} 1 - F(y) - tg(y)
\]

From the first order conditions we obtain:

\[
-f(t) - tg'(t) = 0
\]

which yields

\[
g'(t) = -\frac{1}{2} t
\]

Therefore, we have

\[
g(t) = a - \frac{1}{4} t^2
\]

for some \( a > 0 \). Notice that \( g(\sqrt{6}) \) must be equal to zero (in equilibrium the largest type wins the prize with probability zero). We conclude that

\[
g(t) = \frac{1}{4} 6^2 - \frac{1}{4} t^2
\]

Given \( g \), it is easy to check that the second order conditions are satisfied.