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Game Theory
Solutions to Problem Set 1

1. Consider the following single-person decision problem. The set of available actions is $\{a, b, c\}$. The set of states is $\{\omega_1, \omega_2, \omega_3\}$. The payoffs are given by:

	ω_1	ω_2	ω_3
a	4	1	5
b	5	0	6
c	3	4	3
d	2	9	2

We look for actions that are strictly dominated. First, note the following:

- b is optimal in states ω_1 and ω_3
- d is optimal in state ω_2

Hence, b and d cannot be strictly dominated, which implies that the only potential candidates for strictly dominated actions are a and c.

Then, note that neither a nor c is strictly dominated by any pure strategy, so we have to consider mixed strategies. Consider a mixed strategy $\sigma = (0, p_b, 0, p_d)$, i.e. a mixture of b and d that puts probability p_b on action b, $p_d = (1 - p_b)$ on action d, and probability zero on a and c. For this to strictly dominate action (pure strategy) a, it has to be the case that:

$$\begin{aligned} p_b 5 + (1 - p_b)\, 2 &> 4, \text{ true for } p_b \in \left(\frac{2}{3}, \infty\right) \\ p_b 0 + (1 - p_b)\, 9 &> 1, \text{ true for } p_b \in \left(-\infty, \frac{8}{9}\right) \\ p_b 6 + (1 - p_b)\, 2 &> 5, \text{ true for } p_b \in \left(\frac{3}{4}, \infty\right). \end{aligned}$$

Hence, a is dominated by any mixed-strategy $\sigma = (0, p_b, 0, p_d)$ such that $p_b \in \left(\frac{3}{4}, \frac{8}{9}\right)$ and $p_d = 1 - p_b$.

Similarly, for c to be strictly dominated by a mixed-strategy that puts positive probability only on b and d, it has to be the case that:

$$
\begin{aligned}
p_b 5 + (1-p_b)\,2 &> 3, \text{ true for } p_b \in \left(\frac{1}{3}, \infty\right) \\
p_b 0 + (1-p_b)\,9 &> 4, \text{ true for } p_b \in \left(-\infty, \frac{5}{9}\right) \\
p_b 6 + (1-p_b)\,2 &> 3, \text{ true for } p_b \in \left(\frac{1}{4}, \infty\right)
\end{aligned}
$$

Hence, c is dominated by any mixed-strategy $\sigma = (0, p_b, 0, p_d)$ such that $p_b \in \left(\frac{1}{3}, \frac{5}{9}\right)$ and $p_d = 1 - p_b$.

Together, this implies that a and c, but not b and d, are strictly dominated actions.

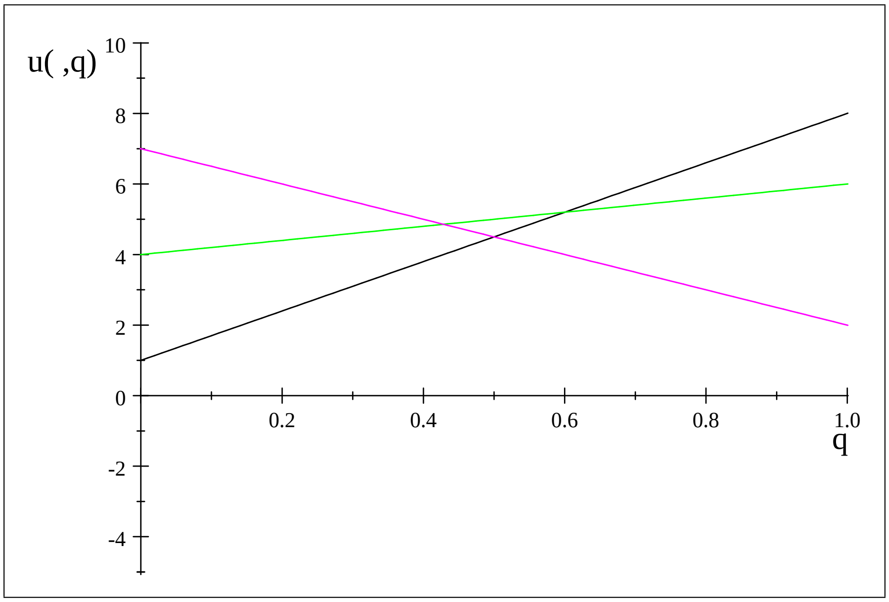
2. A decision maker has three actions (a, b, and c) and faces two states (ω_1 and ω_2). His payoffs are given by:

	ω_1	ω_2
a	8	1
b	6	4
c	2	7

A belief vector is a probability distribution $q = (q(\omega_1),\ q(\omega_2)) \in \triangle(\Omega)$, where $q(\omega_2) = 1 - q(\omega_1)$. The expected utility of the actions a, b and c, given some beliefs q, are:

$$
\begin{aligned}
u(a,q) &= q(\omega_1) \cdot 8 + [1 - q(\omega_1)] \cdot 1 = 7q(\omega_1) + 1 \\
u(b,q) &= q(\omega_1) \cdot 6 + [1 - q(\omega_1)] \cdot 4 = 2q(\omega_1) + 4 \\
u(c,q) &= q(\omega_1) \cdot 2 + [1 - q(\omega_1)] \cdot 7 = -5q(\omega_1) + 7
\end{aligned}
$$

The best-response correspondence can be read directly from a plot of these three functions:



where the two relevant intersections are given by:

$$\begin{aligned} 2q(\omega_1) + 4 &= -5q(\omega_1) + 7 \Rightarrow q(\omega_1) = 3/7 \\ 7q(\omega_1) + 1 &= 2q(\omega_1) + 4 \Rightarrow q(\omega_1) = 3/5. \end{aligned}$$

Formally, the best-response correspondence is:

$$BR(q) = \begin{cases} c, \text{ if } q(\omega_1) \leq 3/7 \\ b, \text{ if } q(\omega_1) \geq 3/7 \text{ and } q(\omega_1) \leq 3/5 \\ a, \text{ if } q(\omega_1) \geq 3/5 \end{cases}$$

Finally, note that extending this problem to allowing for mixed strategies is straight-forward; if we, given some beliefs, have multiple optimal pure strategies, then any mix of those strategies is also optimal.

3. Consider two decision makers with the same set of actions A and the same set of states Ω. The payoff function of the decision maker $i = 1, 2$ is $u_i : A \times \Omega \to \mathbb{R}$. Suppose that for every $a \in A$ and every $\omega \in \Omega$:

$$u_2(a, \omega) = ku_1(a, \omega) + b,$$

where k is a positive number and b a real number. Assume further that the two decision makers have the same beliefs $q \in \Delta(\Omega)$. The expected utility of decision-maker 1 from choosing an action (pure strategy) a, given beliefs q, is:

$$U_1(a, q) \equiv \sum_{\omega} q(\omega) u_1(a, \omega)$$

Decision-maker 2's expected utility from choosing a, given beliefs q, can be written:

$$\begin{aligned}
U_2(a,q) &\equiv \sum_{\omega} q(\omega)u_2(a,\omega) = \sum_{\omega} q(\omega)\left[ku_1(a,\omega)+b\right] \\
&= k\sum_{\omega} q(\omega)u_1(a,\omega) + b\sum_{\omega} q(\omega) \\
&= k\sum_{\omega} q(\omega)u_1(a,\omega) + b \\
&= kU_1(a,q) + b
\end{aligned}$$

Now, given that k is a positive number and b a real number, we have that:

$$\begin{aligned}
U_1(a,q) &\geq U_1(x,q),\ \forall x \in A \\
&\Leftrightarrow \sum_{\omega} q(\omega)u_1(a,\omega) \geq \sum_{\omega} q(\omega)u_1(x,\omega),\ \forall x \in A \\
&\Leftrightarrow k\sum_{\omega} q(\omega)u_1(a,\omega) \geq k\sum_{\omega} q(\omega)u_1(x,\omega),\ \forall x \in A \\
&\Leftrightarrow k\sum_{\omega} q(\omega)u_1(a,\omega) + b \geq k\sum_{\omega} q(\omega)u_1(x,\omega) + b,\ \forall x \in A \\
&\Leftrightarrow U_2(a,q) \geq U_2(x,q),\ \forall x \in A.
\end{aligned}$$

That is, if an action a gives the highest expected utility for one of the decision-makers, it has to give the highest expected utility also for the other decision-maker. Hence, under the assumption that decision-makers are appropriately characterized as expected utility maximizers, we have shown that 1 and 2 will make the same decision.