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Game Theory

Solutions to Problem Set 10

1 Sequential Equilibria

One could approach this problem in different ways. Here, we look for equilibria by exhausting the different possible strategies that player 2 could use. For each possible strategy we check whether there exist equilibria where that strategy is used.

Formally, let σ_2 denote player 2's strategy, where σ_2 is the probability that player 2 plays T. Let $\sigma_1 = (\sigma_1^1,\ \sigma_1^2)$ denote player 1's strategy, where σ_1^1 is the probability that type 1 chooses L, and σ_1^2 is the probability that type 2 chooses L. Finally, let μ denote the probability that player 2 assigns to being at the upper node if player 2 gets to choose (i.e. the beliefs of player 2 in the information set that is reached if player 1 plays L). We divide the different strategies of player 1 into the following three cases:

(i) Assume that player 2 is playing pure strategy T (i.e. assume that $\sigma_2 = 1$). Note that this is optimal for player 2, no matter what beliefs she holds. For player 1, any σ_1 such that $\sigma_1^1 \in [0,1]$ and $\sigma_1^2 = 1$ is a best response to $\sigma_2 = 1$. Hence, there exist a continuum of equilibiria on the following form:

$$\begin{aligned}
\sigma_1 &= (\sigma_1^1,\ \sigma_1^2) \text{ s.t. } \sigma_1^1 \in [0,1],\ \sigma_1^2 = 1 \\
\sigma_2 &= 1 \\
\mu &= \frac{\sigma_1^1}{\sigma_1^1 + \sigma_1^2}
\end{aligned}$$

(ii) Assume that player 2 is playing pure strategy B (i.e. assume that $\sigma_2 = 0$). Note that this is an optimal stretagy for player 2 iff $\mu = 0$. Then, note that the best response of player 1 to $\sigma_2 = 0$ is any strategy $\sigma_1 = (\sigma_1^1,\ \sigma_1^2)$ such that $\sigma_1^1 = 0$ and $\sigma_1^2 \in [0,1]$. If $\sigma_1^2 > 0$, beliefs are determined by Bayes' rule, and must be $\mu = 0$. If $\sigma_1^2 = 0$, we are "free" to assign beliefs. In particular, for beliefs $\mu = 0$, the strategy-beliefs pair constitutes an equilibrium. Hence, we have a continuum of equilibria on the following form:

$$\begin{aligned}
\sigma_1 &= (\sigma_1^1,\ \sigma_1^2) \text{ s.t. } \sigma_1^1 = 0,\ \sigma_1^2 \in [0,1] \\
\sigma_2 &= 0 \\
\mu &= 0
\end{aligned}$$

(iii) Assume that player 2 is playing a fully mixed strategy [i.e. assume that $\sigma_2 \in (0,1)$]. In any such equilibrium, it would have to be the case that type 1 plays R, while type 2 plays L. Given this strategy of player 1, player 2 is indifferent between T and B, so any fully mixed strategy is in fact a best response. Hence, we have a continuum of equilibria on the following form:

$$\begin{aligned} \sigma_1 &= (\sigma_1^1,\ \sigma_1^2) \text{ s.t. } \sigma_1^1 = 0,\ \sigma_1^2 = 1 \\ \sigma_2 &\in (0,1) \\ \mu &= 0 \end{aligned}$$

2 Gibbons, Exercise 4.2

To find the NE of the game, we describe the extensive form given in the problem with the following normal form representation:

	L'	M'	
L	3,0	0,1	q_L
M	0,1	3,0	q_M
R	2,2	2,2	$1-q_L-q_M$
	$p_{L'}$	$1-p_{L'}$	

It is immediate to see that this game does not have a pure strategy NE. Therefore, there does not exist any pure strategy PBE (which follows from the fact that any PBE is also a NE). Also, there does not exist a mixed strategy PBE in which player 2 plays a pure strategy and player 1 a fully mixed strategy. To see this, note that each pure strategy of player 2 has pure strategy best response of player 1.

Together, this implies that in any PBE, player 2 must be randomizing between L' and M'. Hence, in any PBE, player 2 must be indifferent between playing L' and M'. This implies that player 1 must be playing L and M with equal probability (because of the payoff structure).

Now, if $p_L = p_M > 0$, for player 1 to want to randomize, it must be the case that $p_{L'} = 1/2$. However, if $p_{L'} = 1/2$, the expected payoff of L (or M) is 1.5<2, so player 1 could profitably deviate to play R. Therefore, the only candidate for a PBE is when $p_L = p_M = 0$. That is, the only candidate for a PBE is when 1 plays pure strategy R, and 2 randomizes between L' and R' in such a way that it is in fact optimal for 1 to play R. This is the case when:

$$\begin{aligned} 2 &\geq 3p_{L'} \Rightarrow \frac{2}{3} \geq p_{L'} \\ 2 &\geq 3p_{R'} \Rightarrow \frac{2}{3} \geq p_{R'} \end{aligned}$$

Hence, the PBE in the game are described by:

$$\begin{aligned} \sigma_1 &= (q_L,\ q_M,\ 1-q_L-q_M) = (0,0,1) \\ \sigma_2 &= (p_{L'},\ 1-p_{L'}) \text{ s.t. } p_{L'} \in [1/3,\ 2/3] \\ \mu &= 1/2 \end{aligned}$$

where μ is the probability that player 2 assigns to being at the left node is 2's information set is reached.

3 Gibbons, Exercise 4.5

(a) Since we restrict to pure strategies, the strategy sets are the following:

$$
\begin{aligned}
S_1 &= \{(L,L),\ (L,R),\ (R,L),\ (R,R)\} \\
S_2 &= \{(u,u),\ (u,d),\ (d,u),\ (d,d)\}
\end{aligned}
$$

where a strategy s$_1 = (x,\ y)$ indicates that type 1 of player 1 chooses x and type 2 choses y, and s$_2 = (w,\ z)$ indicates that player 2 chooses action w at the left information set and z and the right information set.

In order to rule out most strategy profiles as possible PBE, we can look at the best response correspondences. The following two tables illustrate:

s_1		$BR_2(s_1)$	
type 1	type 2	left set	right set
L	L	u	u/d
L	R	u	d
R	L	d	u
R	R	u/d	d

s_2		$BR_1(s_2)$	
left set	right set	type 1	type 2
u	u	L	R
u	d	R	R
d	u	L	L
d	d	R	L

$>$From these best responses, we see that $s = (s_1, s_2) = ((R,R),(u,d))$ is the only possible PBE strategy profile. To fully specify the PBE, we also have to specify the equilibrium beliefs. Let $\mu = (\mu_L, \mu R)$ denot player 2's beliefs, where μ_L indicate the probability that 2 assigns to being at the upper node given that the left information set is reached, and μ_R indicate the probability that 2 assigns to being at the upper node given that the right information set is reached. Note that, given $s_1 = (R,R)$, Bayes' rule pins down μ_R $(= .5)$ but not μ_L. Hence, we are "free" to specify beliefs at the left information set in such a way that (u,d) is in fact optimal. This is the case when:

$$
\begin{aligned}
2\mu_L &\geq (1-\mu_L)1 \\
&\Rightarrow \mu_L \geq \frac{1}{3}.
\end{aligned}
$$

Together, this implies that any strategy-belief pair in which

$$
\begin{aligned}
s &= (s_1, s_2) = ((R,R),(u,d)) \\
\mu &= (\mu_L, \mu_R) \text{ s.t. } \mu_R = .5,\ \mu_L \geq 1/3
\end{aligned}
$$

constitute a PBE.

(b) In this question, the strategy sets are the following:

$$
\begin{aligned}
S_1 &= \{(x,y,z) \mid x,y,x \in \{L,R\}\} \\
S_2 &= \{(u,u),\ (u,d),\ (d,u),\ (d,d)\}
\end{aligned}
$$

where a strategy $s_1 = (x,\ y, z)$ indicates that type 1 of player 1 chooses x, type 2 choses y and type 3 chooses z. For player 2, $s_2 = (w,\ z)$ indicates that 2 chooses action w at the left information set and z and the right information set.

Now, note that sequential rationality implies that, in any PBE, player 2's strategy have to specify u at the left information set. This implies that we can limit our consideration to strategy profiles in which $s_2 \in \{(u, u),\ (u, d)\}$. Now, by looking at the best response structure, we can identify the "PBE candidates":

s_2		$BR_1(s_2)$			$BR_2(BR_1(s_2))$	
left set	right set	type 1	type 2	type 3	left set	right set
u	u	L	L	L	u	u/d
u	d	L	L	R	u	d

Hence, we have two strategy profiles that are PBE candidates: $s' = ((L, L, L),\ (u, u))$ and $s'' = ((L, L, R),\ (u, d))$. In order to fully specify the equilibria, we must specigy the beliefs in each equilibrium. Let $\mu = [\mu_L,\ \mu_R] = \left[\left(\mu_L^1, \mu_L^2, \mu_L^3\right),\ \left(\mu_R^1, \mu_R^2, \mu_R^3\right)\right]$ denote player 2's beliefs, where μ_x^a indicates the probability that player 2 assigns to player 1 being of type a, when player 2 is called to move at information set x.

Now, note that in any PBE in which strategy profile s" is played, the beliefs of player 2 are fully pinned down by Bayes' rule. However, stretegy profile s' does not pin down the equilibrium beliefs at the right information set, since given the strategy $s_1 = ((L, L, L)$ this information set is not reached along the equilibrium path. We know that any such PBE is has to be the case that $s_2 = (u, u)$ is optimal for player 2. Focusing on the right information set, this implies that beliefs must be such that:

$$\begin{aligned} \mu_R^1 + \mu_R^2 &\geq 1 - \mu_R^1 + \mu_R^2 \\ &\Rightarrow \mu_R^1 + \mu_R^2 \geq \frac{1}{2} \end{aligned}$$

Together, this implies that any strategy-belief pair in which

$$\begin{aligned} s' &= (s_1, s_2) = ((L, L, L),\ (u, u)) \\ \mu &= [\mu_L,\ \mu_R] = \left[\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right),\ \left(\mu_R^1, \mu_R^2, \mu_R^3\right)\right] \text{ s.t. } \mu_R^1 + \mu_R^2 \geq \frac{1}{2},\ \mu_R^3 = 1 - \mu_R^1 - \mu_R^2 \end{aligned}$$

constitute a PBE.

In addition, we have a separating PBE in which:

$$\begin{aligned} s'' &= ((L, L, R),\ (u, d)) \\ \mu &= [\mu_L,\ \mu_R] = \left[\left(\frac{1}{2}, \frac{1}{2}, 0\right),\ (0, 0, 1)\right] \end{aligned}$$