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Game Theory
Solutions to Problem Set 2

1 Find all NE

(a)

	L	R
U	$3,4$	$-2,6$
D	$0,3$	$-5,1$

First, note that pure strategy D is strictly dominated by U $\Rightarrow$we can eliminate D. Then pure strategy L is strictly dominated by R $\Rightarrow$w e can eliminate L.Hence, the unique NE is (U, R).

(b)

	L	R
U	$4,5$	$3,1$
D	$4,0$	$0,6$

First, note that there is a pure strategy NE: (U, L). To find mixed strategy NE, suppose player 1 (row player) is playing a mixed strategy $\sigma_1^\alpha = (\alpha, 1-\alpha)$, while player 2 is playing mixed strategy $\sigma_2^\beta = (\beta, 1-\beta)$. For this to be a fully mixed NE:

$$\begin{aligned} 4\beta + 3(1-\beta) &= 4\beta \\ 5\alpha &= 1\alpha + 6(1-\alpha) \end{aligned}$$

$\Rightarrow$ $\alpha = \frac{3}{5}$, $\beta = 1$. But then $\beta = 1$ implies that no equilibrium can be fully mixed. [We could have concluded this right away by noting that D is weakly dominated.]

Can there be a partially mixed NE? If so, it must be player 1 who is playing a full support strategy, while player 2 is playing L. All we need to check then is that player 2 would want to play L, which is the case when

$$5\alpha \geq 1\alpha + 6(1-\alpha)$$

i.e. when $\alpha \in [\frac{3}{5}, 1]$. To summarize, the set of NE is

$$\left\{ \left[\sigma_1^\alpha = (\alpha, 1-\alpha),\ \sigma_2^\beta = (\beta, 1-\beta) \right] \mid \alpha \in [\frac{3}{5}, 1],\ \beta = 1 \right\}$$

(c)

$$\begin{array}{cccccc} & L & C & R & \\ U & 6,6 & 1,2 & 3,3 & \alpha \\ M & 2,1 & 4,7 & 4,3 & 1-\alpha \\ D & 3,4 & 2,5 & 3,9 & \\ & \beta & 1-\beta & & \end{array}$$

First, note that the pure strategy D is strictly dominated by any mixed strategy $\sigma_1^\alpha = (\alpha, 1-\alpha, 0)$ s.t. $\frac{1}{4} < \alpha < \frac{2}{3} \Rightarrow$ we can eliminate D. Then,

pure strategy R is strictly dominated by any mix strategy $\sigma_2^\beta = (\beta, 1-\beta, 0)$ s.t. $\frac{1}{4} < \beta < \frac{2}{3} \Rightarrow$ we can eliminate R. Hence, we are left with the following game:

$$\begin{array}{cccc} & L & C & \\ U & 6,6 & 1,2 & \alpha \\ M & 2,1 & 4,7 & 1-\alpha \\ & \beta & 1-\beta & \end{array}$$

By inspection, this game has two pure strategy NE: (U, L) and (C,M). There are no partially mixed NE (since all best-responses to pure strategies are unique). Any fully mixed NE would have to satisfy:

$$\begin{aligned} 6\beta + 1(1-\beta) &= 2\beta + 4(1-\beta) \\ 6\alpha + 1(1-\alpha) &= 2\alpha + 7(1-\alpha) \end{aligned}$$

$\Rightarrow \ \beta = \frac{3}{7}$, $\alpha = \frac{3}{5}$. Hence, the set of NE can be described by the following (α, β) pairs: $\left\{(1,1), (0,0), (\frac{3}{5}, \frac{3}{7})\right\}$.

Finally, note that we could also have chosen to work with the best-response correspondences in the reduced (2x2) game, to find the set of NE. These best-response correspondences look the following:

$$BR_1(\sigma_2^\beta) = \left\{ \begin{array}{l} \alpha = 0, \ \beta < \frac{3}{7} \\ \alpha \in [0,1], \ \beta = \frac{3}{7} \\ \alpha = 1, \ \ \beta > \frac{3}{7} \end{array} \right. , \ BR_2(\sigma_1^\alpha) = \left\{ \begin{array}{l} \beta = 0, \ \alpha < 0.6 \\ \beta \in [0,1], \ \alpha = \frac{3}{5} \\ \beta = 1, \ \alpha > 0.6 \end{array} \right.$$

and in order to find the NE, we would have to look for a fixed-point with these correspondences.

2 Divide the dollar

The best-response correspondence for player $i \in \{1, \ 2\}$ is:

$$BR_i(s_j) = \left\{ \begin{array}{l} 1 - s_j, \ \text{if } s_j \in [0,1) \\ s_i \in [0, \infty], \ \text{if } s_j \in [1, \infty] \end{array} \right.$$

where $j \in \{1,\ 2\}\,,\quad j \neq i$. The set of NE is:

$$\left\{(s_1, s_2) \in R_+^2 \mid (s_1 \in [0,1] \text{ and } s_2 = 1 - s_1) \text{ or } (s_1 \in [1,\infty] \text{ and } s_2 \in [1,\infty])\right\}$$

There are NE in weakly dominated strategies. These are the NE where a player i is playing a strategy $s_i \geq 1$, and the NE where a player is is playing a strategy $s_i = 0$. To see this, note that playing a strategy greater then 1 will yield payoff of zero whatever the strategy of the opponent, while playing strategy which is less than or equal to one could yield a positive payoff. Similarly, offering zero would always yield zero payoff (against all possible plays of the other player), while offering a stricly positive number (less than one) could yield a positive payoff.

3 Rock-Scissors-Paper

3.1 Approach 1

	R	S	P	
R	$0,0$	$1,-1$	$-1,1$	α
S	$-1,1$	$0,0$	$1,-1$	β
P	$1,-1$	$-1,1$	$0,0$	$1-\alpha-\beta$
	p	q	$1-q-p$	

Note first that there are no pure NE in this game (it should be intuitively clear why that is). To find the mixed NE, suppose player 1 is playing $\sigma^{\alpha\beta} = (\alpha, \beta, 1-\alpha-\beta)$, and player 2 is playing $\sigma^{pq} = (p, q, 1-p-q)$. Define the following value functions:

$$\begin{aligned}
V_1(R, \sigma^{pq}) &= q - 1 + q + p = 2q + p - 1 \\
V_1(S, \sigma^{pq}) &= -p + 1 - p - q = 1 - 2p - q \\
V_1(P, \sigma^{pq}) &= p - q
\end{aligned}$$

For player 1 we have:

$$\begin{aligned}
V_1(R, \sigma^{pq}) &\geq V_1(S, \sigma^{pq}) \Rightarrow 2q + p - 1 \geq 1 - 2p - q \Rightarrow (q + p) \geq \frac{2}{3} \\
V_1(R, \sigma^{pq}) &\geq V_1(P, \sigma^{pq}) \Rightarrow 2q + p - 1 \geq p - q \Rightarrow q \geq \frac{1}{3} \\
V_1(S, \sigma^{pq}) &\geq V_1(P, \sigma^{pq}) \Rightarrow 1 - 2p - q \geq p - q \Rightarrow p \leq \frac{1}{3}
\end{aligned}$$

which gives us the following best-response:

$$BR_1(\sigma^{pq}) = \begin{cases} \alpha = 1 \text{ if } q > \frac{1}{3}, (p+q) > \frac{2}{3} \\ \beta = 1 \text{ if } p < \frac{1}{3}, (p+q) < \frac{2}{3} \\ \alpha = \beta = 0 \text{ if } q < \frac{1}{3}, p > \frac{1}{3} \\ \alpha, \beta \in [0,1], \alpha + \beta = 1 \;\text{ if } q > \frac{1}{3}, (p+q) = \frac{2}{3} \\ \alpha \in [0,1], \beta = 0 \text{ if } q = \frac{1}{3}, (p+q) > \frac{2}{3} \\ \alpha = 0, \beta \in [0,1] \;\text{ if } p = \frac{1}{3}, (p+q) < \frac{2}{3} \\ \alpha, \beta \in [0,1], \alpha + \beta < 1 \;\text{ if } p = q = \frac{1}{3} \end{cases}$$

Since the game is symmetric:

$$BR_2(\sigma^{\alpha\beta}) = \begin{cases} p = 1 \text{ if } \alpha > \frac{1}{3}, (\alpha+\beta) > \frac{2}{3} \\ q = 1 \text{ if } p < \frac{1}{3}, (\alpha+\beta) < \frac{2}{3} \\ p = q = 0 \text{ if } \beta < \frac{1}{3}, \alpha > \frac{1}{3} \\ p, q \in [0,1], p + q = 1 \;\text{ if } \beta > \frac{1}{3}, (\alpha+\beta) = \frac{2}{3} \\ p \in [0,1], q = 0 \text{ if } \beta = \frac{1}{3}, (\alpha+\beta) > \frac{2}{3} \\ p = 0, q \in [0,1] \;\text{ if } \alpha = \frac{1}{3}, (\alpha+\beta) < \frac{2}{3} \\ p, q \in [0,1], p + q < 1 \;\text{ if } \alpha = \beta = \frac{1}{3} \end{cases}$$

Using these best-responses, we can find a unique NE in which $(\alpha, \beta, 1-\alpha-\beta) = (p, q, 1-p-q) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

3.2 Approach 2

The procedure for finding the NE given above is quite tedious, and there is a simpler approach. Here, we first note that there is no NE in pure strategies. Hence, we have to look for mixed strategy NE. Suppose there is a fully mixed NE. Then, in that NE the following must be true for player 1:

$$\begin{aligned} V_1(S, \sigma^{pq}) &= V_1(P, \sigma^{pq}) \Rightarrow 1 - 2p - q = p - q \Rightarrow p = \frac{1}{3} \\ V_1(R, \sigma^{pq}) &= V_1(P, \sigma^{pq}) \Rightarrow 2q + p - 1 = p - q \Rightarrow q = \frac{1}{3} \end{aligned}$$

which implies that $(p, q, 1-p-q) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. And by symmetry, this implies that $(\alpha, \beta, 1-\alpha-\beta) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Hence, we have found a symmetric mixed strategy NE where the players are randomizing over all three actions. By the uniqueness of the solution to these equations, it also follows that this is the only fully mixed NE.

To show that this NE is unique, all we need to do then is to show that there cannot be a NE in which one of the players play a strategy where the support consists of exactly two actions. Suppose this is the case; in particular, suppose we have a NE in which player 2 randomizes over S and P. Then, player 1 cannot be playing P with positive probability. Hence, player 1 must be randomizing over R and S (since we have already argued that there is no NE in which any player plays a pure strategy). But if player 1 is playing both R and S with positive probability, player 2 cannot be playing P with positive probability - i.e. we have a contradiction.

We can repeat the same argument for any NE in which player 2 randomizes over S and R, and R and P. Finally, the symmetry of the setup delivers the same result for player 1, which implies that there cannot be a NE where one of the players randomizes over exactly two strategies. Hence, the equilibrium we found above is unique.

4 Increasing Linear Transformation

A profile of mixed strategies σ is a NE iff

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \quad \textit{for all } i \textit{ and } \sigma_i' \in \triangle(S_i)$$

Then, all we need to note is that

$$\begin{aligned}
u_i(\sigma_i, \sigma_{-i}) &\geq u_i(\sigma_i', \sigma_{-i}) \\
&\Leftrightarrow \sum_{s_i \in Supp(\sigma_i)} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \geq \sum_{s_i' \in Supp(\sigma_i')} \sigma_i'(s_i') u_i(s_i', \sigma_{-i}) \\
&\Leftrightarrow A_i \sum_{s_i \in Supp(\sigma_i)} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) + B_i \geq A_i \sum_{s_i' \in Supp(\sigma_i')} \sigma_i'(s_i') u_i(s_i', \sigma_{-i}) + B_i \\
&\Leftrightarrow \sum_{s_i \in Supp(\sigma_i)} \sigma_i(s_i) [A_i u_i(s_i, \sigma_{-i}) + B_i] \geq \sum_{s_i' \in Supp(\sigma_i')} \sigma_i'(s_i') [A_i u_i(s_i', \sigma_{-i}) + B_i] \\
&\Leftrightarrow \sum_{s_i \in Supp(\sigma_i)} \sigma_i(s_i) \tilde{u}_i(s_i, \sigma_{-i}) \geq \sum_{s_i' \in Supp(\sigma_i')} \sigma_i'(s_i') \tilde{u}_i(s_i', \sigma_{-i}) \\
&\Leftrightarrow \tilde{u}_i(\sigma_i, \sigma_{-i}) \geq \tilde{u}_i(\sigma_i', \sigma_{-i}) \ \blacksquare
\end{aligned}$$

5 Elimination of Strictly Dominated

$(\Longrightarrow)$

Suppose σ is a NE of G. That is,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \ \forall i, \ \sigma_i' \in \triangle(S_i).$$

We know from theory/lectures that the support of a NE strategies in G could not contain strictly dominated strategies, which implies that $Supp(\sigma) \in S^1$. Now, suppose σ is not a NE in the game G^1.Then

$$\exists i, \ \sigma_i' \in \triangle(S_i^1) \subset \triangle(S_i) \ \ s.t. \ \ u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i})$$

which is a contradiction.

$(\Longleftarrow)$

Suppose σ is a NE of G^1. That is,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}),\ \forall i,\ \sigma_i' \in \triangle(S_i^1)$$

Suppose σ is not a NE of G. Then,

$$\begin{aligned}
\exists i,\ \tilde{\sigma}_i &\in \triangle(S_i)\ \ s.t.\ \ u_i(\tilde{\sigma}_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}) \\
&\Rightarrow u_i(\tilde{\sigma}_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}),\ \forall \sigma_i' \in \triangle(S_i^1) \subset \triangle(S_i) \\
&\Rightarrow \sum_{s_i \in Supp(\tilde{\sigma}_i)} \tilde{\sigma}_i(s_i) u_i(s_i, \sigma_{-i}) > \sum_{s_i' \in Supp(\sigma_i')} \sigma_i'(s_i) u_i(s_i, \sigma_{-i}),\ \forall \sigma_i' \in \triangle(S_i^1) \subset \triangle(S_i) \\
&\Rightarrow \exists \tilde{s}_i \in S_i \backslash S_i^1\ \ s.t.\ \ \tilde{s}_i \in Supp(\tilde{\sigma}_i)\ \ and\ \ u_i(\tilde{s}_i, \sigma_{-i}) > u_i(s_i, \sigma_{-i}),\ \forall s_i \in S_i^1 \subset S_i
\end{aligned}$$

Then, noting that any strictly dominated strategy have to be strictly dominated by a strategy in S_i^1, $\tilde{s}_i$ should be in S_i^1, which is a contradiction. ■