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Game Theory

Solutions to Problem Set 3

1 Cournot Oligopoly

Strategies set for player i:

$$S_i = q_i \in [0, \infty)$$

Payoffs,

$$\Pi_i = \begin{cases} (a - b\sum_{i=1}^{n} q_i)q_i - cq_i & \text{if } (a - b\sum_{i=1}^{n} q_i) > 0 \\ 0 & \text{Otherwise} \end{cases}$$

Suppose that $p = (a - b\sum_{i=1}^{n} q_i) > 0$, then each player i maximizes her payoff:

$$\begin{aligned} & \max_{q_i}(a - b\sum_{i=1}^{n} q_i)q_i - cq_i \\ FOC &: \quad a - b\sum_{j \neq i}^{n} q_j^* - 2bq_i - c = 0 \\ q_i^* &= \frac{a - b\sum_{j \neq i}^{n} q_j^* - c}{2b} \end{aligned}$$

Given that we are focusing on symmetric NE, we can impose $q_i^* = q^*$ for some q^* satisfying:

$$\begin{aligned} 2bq^* + b(n-1)q^* &= (a - c) \\ q^* &= \frac{(a-c)}{b(n+1)} \\ Q &\equiv \sum_i q_i^* = \frac{n(a-c)}{b(n+1)} \end{aligned}$$

Here, we need to check that indeed $p = (a - b\sum_{i=1}^{n} q_i) > 0$:

$$
\begin{aligned}
p &= (a - b\sum_{i=1}^{n} q_i) = \\
&= (a - b\frac{n(a-c)}{b(n+1)}) \\
&= \frac{a(n+1) - n(a-c)}{(n+1)} \\
&= \frac{a+nc}{(n+1)} > 0 \quad \checkmark
\end{aligned}
$$

Hence, the symmetric NE is

$$q_i^* = q^* = \frac{(a-c)}{b(n+1)} \ \forall i$$

As $n \longrightarrow \infty$

$$
\begin{aligned}
\lim_{n\to\infty} \sum_i q_i^* &= \lim_{n\to\infty} \frac{n(a-c)}{b(n+1)} = \frac{(a-c)}{b} \\
&\Rightarrow \ p = (a - b\sum_{i=1}^{n} q_i) = c \\
&\Rightarrow \ \Pi_i = 0
\end{aligned}
$$

As $n \longrightarrow \infty$, we approach the competitive equilibrium.

2 Cournot Oligopoly

In the same way as in the previous question, we find:

$$
\begin{aligned}
q_1^* &= \frac{a - bq_2^* - c_1}{2b} = \frac{a-c_1}{2b} - \frac{1}{2}q_2^* \\
q_2^* &= \frac{a - bq_1^* - c_2}{2b} = \frac{a-c_2}{2b} - \frac{1}{2}q_1^*
\end{aligned}
$$

This is a system of two equations in two unknows. Solve this system to find the NE:

$$
\begin{aligned}
q_1^* &= \frac{a-c_1}{2b} - \frac{1}{2}\left[\frac{a-c_2}{2b} - \frac{1}{2}q_1^*\right] \\
\frac{3}{4}q_1^* &= \frac{a}{4b} - \frac{c_1}{2b} + \frac{c_2}{4b} \\
&\Rightarrow \; q_1^* = \frac{a}{3b} - \frac{2c_1}{3b} + \frac{c_2}{3b} \\
&= \frac{1}{3b}\left(a - 2c_1 + c_2\right) \\
&\Rightarrow \; q_2^* = \frac{a-c_2}{2b} - \frac{1}{6b}\left(a - 2c_1 + c_2\right) \\
&= \frac{1}{3b}\left(a + c_1 - 2c_2\right)
\end{aligned}
$$

Knowing that $0 < c_1 < c_2$ we have that

$$q_1^* = \frac{1}{3b}\left(a - 2c_1 + c_2\right) > \frac{1}{3b}\left(a + c_1 - 2c_2\right) = q_2^*.$$

And since both firms are facing the same prince

$$
\begin{aligned}
p &= a - b\left(q_1^* + q_2^*\right) \\
&= a - \frac{1}{3}\left(a - 2c_1 + c_2 + a + c_1 - 2c_2\right) \\
&= \frac{1}{3}\left(a + c_1 + c_2\right) > 0
\end{aligned}
$$

we can show that the profit is greater for firm 1:

$$\left(p - c_1\right) q_1^* > \left(p - c_2\right) q_2^*$$

3 Bertrand Oligopoly

Strategies for player i:

$$S_i = p_i \in [0, \infty)$$

$$Payoffs : \Pi_i(p_i, p_j) = \begin{cases} (a - bp_i)(p_i - c) & \text{if } p_i < p_j \\ \frac{1}{2}(a - bp_i)(p_i - c) & \text{if } p_i = p_j \\ 0 & \text{Otherwise} \end{cases}$$

Notice that, in equilibrium, no firm will suggest a price $p < c$ or $p > \frac{a}{b}$. Thus, we can restrict attention to:

$$S_i = p_i \in [c, \frac{a}{b}]$$

Now consider the following cases:

$$Case\ 1: \ c < p_i < p_j \leq \frac{a}{b}$$

In this case, firm i capture the entire market. However, since $c < p_i$, then firm j has the incentive to decrease it's price such that $c < p_j < p_i$. It would then capture the entire market a have a strictly positive profit. And using the same argument, there could not be a NE where $c < p_j < p_i \leq \frac{a}{b}$.

$$Case\ 2: \ c = p_i < p_j \leq \frac{a}{b}$$

In this case, holding p_j constant, firm i has the incentive to increase p_i.

$$Case\ 3: \ c < p_i = p_j = p < \frac{a}{b}$$

In this case, both firms share the market demand, as is indicated in $\Pi_i(p_i, p_j)$. In this case, each firm has incentive to lower p by an arbitrary small amount $\varepsilon > 0$ (say to p') and thus capture the entire market demand. Since the market demand function is continuous, when ε is small enough, $(a - bp')(p' - c) > \frac{1}{2}(a - bp)(p - c)$.

$$Case\ 4: \ p_i = p_j = p = \frac{a}{b}$$

Setting $p = \frac{a}{b}$, $\Pi(p) = 0$ and thus each firm has the obvious incentive to decrease its price, as $(a - bp')(p' - c) > 0$ for $c < p' < \frac{a}{b}$.

$$Case\ 5: \ p_i = p_j = p = c$$

Here, both firms have profits $\Pi(c) = 0$ and there is no incentive to deviate, since increasing the price, holding the other players price constant, will still yield 0 profits.

Together, this implies that the unique pure strategy NE is (c, c).

4 Bertrand Oligopoly with Fixed Costs

4.1 Firm 1 has fixed costs F>0

Suppose firm 2 is a monopolist. Then firm 2 faces the following optimization problem:

$$\max_p (p - c)(a - bp).$$

The solution to the above problem is $p_m = \frac{a+bc}{2b}$. The profits of firm 2 when it charges the price p_m are $\pi_m = \frac{(a-bc)^2}{4b}$.

There are two cases to consider.

(i) $F > \frac{(a-bc)^2}{4b}$. The pure-strategy Nash equilibria are:

$$\begin{array}{ll} (p_1 = p, p_2 = p) & \text{where } p \in [c, p_m] \\ (p_1, p_2 = p_m) & \text{for } p_1 > p_m. \end{array}$$

(ii) $F \leqslant \frac{(a-bc)^2}{4b}$. Let $\hat{p}$ be defined by:

$$\begin{array}{c} (\hat{p} - c)(a - b\hat{p}) = F \\ \hat{p} < p_m. \end{array}$$

That is, $\hat{p}$ is the smallest price at which the profits of firm 2 (when it is a monopolist) are equal to firm 1's fixed cost F. It is easy to check that

$$\hat{p} = \frac{a + bc - \sqrt{(a-bc)^2 - 4bF}}{2b}.$$

The pure-strategy Nash equilibria are:

$$(p_1 = p, p_2 = p) \quad \text{where } p \in [c, \hat{p}].$$

4.2 Both firms have fixed costs F>0

We need to consider three different cases.

(i) $F > \frac{(a-bc)^2}{4b}$. The pure-strategy Nash equilibria are:

$$(p_1, p_2) \quad \text{where } p_1 \geqslant \tfrac{a}{b}, p_2 \geqslant \tfrac{a}{b}.$$

(ii) $F < \frac{(a-bc)^2}{4b}$.The pure-strategy Nash equilibrium is:

$$(p_1 = \hat{p}, p_2 = \hat{p}),$$

where $\hat{p}$ is defined in part (a).

(iii) $F = \frac{(a-bc)^2}{4b}$.The pure-strategy Nash equilibria are:

$$\begin{array}{ll} (p_1 = p_m, p_2) & \text{with } p_2 \geqslant p_m \\ (p_1, p_2 = p_m) & \text{with } p_1 \geqslant p_m \\ (p_1, p_2) & \text{where } p_1 \geqslant \frac{a}{b}, p_2 \geqslant \frac{a}{b}, \end{array}$$

where $p_m = \frac{a+bc}{2b}$.

4.3 Different marginal costs

We have to consider two cases.

(i) $c_2 < p_m = \frac{a+bc}{2b}$. The pure-strategy Nash equilibria are:

$$(p_1 = p, p_2 = p) \quad \text{where } p \in [c_1, c_2].$$

(ii) $c_2 \geqslant p_m$. The pure-strategy Nash equilibria are:

$$\begin{array}{ll} (p_1 = p, p_2 = p) & \text{where } p \in [c_1, p_m] \\ (p_1 = p_m, p_2) & \text{with } p_2 \geqslant p_m. \end{array}$$

Finally, note that the result in this question (as in parts a and b) depends on the assumption that one of the firms capture the entire market when they charge the same price. Clearly, this is not an entire realistic feature of the model, and a good exercise would be to investigate how the results change when this assumption is changed in some way.

5 Bertrand Oligopoly with "Meeting"

Market demand is:

$$q = a - bp$$

Set of strategies:

$$S_i = R_+ \times \{Y, N\}$$

We compute the monopoly price:

$$\begin{aligned} & \max_p (a - bp)(p - c) \\ FOC &: \quad a - 2bp + bc = 0 \\ p^M &= \frac{a}{2b} + \frac{1}{2}c \end{aligned}$$

Suppose $S_2 = p_2 \times Y$

$$BR_1(p_2^*, Y) = \begin{cases} if\ p_2^* > p^M \Rightarrow (p_1^* = p^M, \{Y, N\}) \\ if\ p_2^* = p^M \Rightarrow (p_1^* = p^M, \{Y, N\}), (p_1^* > p^M, Y) \\ if\ c < p_2^* < p^M \Rightarrow (p_1^* \in [p_2^*, \infty), Y), (p_1^* = p_2^*, N) \\ if\ p_2^* = c \Rightarrow (p_1^* \in [c, \infty), \{Y, N\}) \end{cases}$$

$$BR_1(p_2^*, N) = \begin{cases} if\ p_2^* > p^M \Rightarrow (p_1^* = p^M, \{Y, N\}) \\ if\ p_2^* = p^M \Rightarrow (p_1^* < p^M, \{Y, N\}) \\ if\ c < p_2^* < p^M \Rightarrow (p_1^* < p_2^*, \{Y, N\}) \\ if\ p_2^* = c \Rightarrow (p_1^* \in [c, \infty), \{Y, N\}) \end{cases}$$

Since the game is symmetric, then:

$$BR_2(p_1^*, Y) = \begin{cases} if\ p_1^* > p^M \Rightarrow (p_2^* = p^M, \{Y, N\}) \\ if\ p_1^* = p^M \Rightarrow (p_2^* = p^M, \{Y, N\}), (p_2^* > p^M, Y) \\ if\ c < p_1^* < p^M \Rightarrow (p_2^* \in [p_2^*, \infty), Y), (p_2^* = p_1^*, N) \\ if\ p_1^* = c \Rightarrow (p_2^* \in [c, \infty), \{Y, N\}) \end{cases}$$

$$BR_2(p_1^*, N) = \begin{cases} if\ p_1^* > p^M \Rightarrow (p_2^* = p^M, \{Y, N\}) \\ if\ p_1^* = p^M \Rightarrow (p_2^* < p^M, \{Y, N\}) \\ if\ c < p_1^* < p^M \Rightarrow (p_2^* < p_1^*, \{Y, N\}) \\ if\ p_1^* = c \Rightarrow (p_2^* \in [c, \infty), \{Y, N\}) \end{cases}$$

Now, we can find the set of NE directly by looking for a fixed point of these best-responses. Alternatively, we follow the approach used in questions 3-4, i.e. to divide the set of possible strategy pairs into a limited number of different "cases," and for each case evaluate whether we can have a NE of that particular form. This is more tedious than in questions 3-4, but in principle not difficult. Either way, we get the following four different types of equilibria:

NE type 1: $\{(p, Y), (p, Y)$ such that $p \in [c, p^M]\}$
NE type 2: $\{(p^M, Y), (p_2 > p^M, Y)\}$
NE type 3: $\{(p_1 > p^M, Y), (p_2, Y)\}$
NE type 4 $\{(c, \{Y, N\}), (c, \{Y, N\})\}$

6 First-Price Auction

Set of players is $\{1, ..., n\}$. Each player i submit a bid $b_i \in [0, \infty)$. Without loss of generality, we can restrict attention to bids $b_i \in [0, v]$.

To construct the payoff matrix, define the following bid:

$$b_{-i}^* = \max_{j \neq i} b_j$$

The payoffs are :

$$u_i(b_i, b_{-i}^*) = \begin{cases} v - b_i, \quad \text{if } b_i > b_{-i}^* \\ \frac{1}{k+1}(v - b_i), \quad \text{if } b_i = b_{-i}^* \text{ and there are } k \text{ winners} \\ 0, \quad \text{otherwise} \end{cases}$$

Suppose that player i is the only winner of the auction, i.e., $b_i > b_{-i}^*$. If $b_i < v$, then all other players have incentive to bid either b_i or $(b_i + \varepsilon)$, so it cannot be a NE. If $b_i = v$, then player i has incentive to lower her bid s.t. it would still be $b_i > b_{-i}^*$. But we just argued that such a scenario could not be a NE. Thus, if a NE exists, it has to be that we have more than one winner.

Suppose the winning bid is $b_i = b_{-i}^*$, and there are $(k + 1)$ winners and $(k + 1) \geq 2$, then the winner's payoff is:

$$u_i(b_i, b_{-i}^*) = \frac{1}{k+1}(v - b_i)$$

If $b_i = b_{-i}^* < v$, then every player has incentive to bid $b_i < b < v$ and thus this could not be NE.

If $b_i = b^*_{-i} < v$, then every player has incentive to bid $b_i < b < v$. Hence, this could not be a NE. If $b_i = b^*_{-i} = v$, then no player has incentive to deviate. Increasing ones bid s.t. $b \leq v$ would not increase the payoff (it would remain at zero). Bidding $b > v$ would yield a negative payoff. And if one of the winning players bid less that v, then she does not gain from this deviation.

$$NE : b_i = b^*_{-i} = v \text{ with at least 2 winners.}$$