Collegio Carlo Alberto

Game Theory

Solutions to Problem Set 4

1 Hotelling's model

1.1 Two vendors

Consider a strategy profile (s_1, s_2) with $s_1 \neq s_2$. Suppose $s_1 < s_2$. In this case, it is profitable to for player 1 to deviate and choose a location $s_1' \in (s_1, s_2)$. To see this, note that

$$u_1(s_1', s_2) = \frac{s_1' + s_2}{2} > \frac{s_1 + s_2}{2} = u_1(s_1, s_2).$$

Thus, in a pure-strategy Nash equilibrium both players choose the same location. Consider now the profile $(s_1 = s, s_2 = s)$ where $s \neq 1/2$. In this case, both players get $1/2$. However, if a player deviates and chooses $1/2$, her payoff is strictly greater than $1/2$. So, we are left with $(s_1 = 1/2, s_2 = 1/2)$. In this case, no player has an incentive to deviate:

$$\begin{aligned} u_i\left(s_i = \frac{1}{2}, s_j = \frac{1}{2}\right) &= \frac{1}{2} > \frac{s_i' + \frac{1}{2}}{2} = u_i\left(s_i', s_j = \frac{1}{2}\right), \text{ where } s_i' < \frac{1}{2} \\ u_i\left(s_i = \frac{1}{2}, s_j = \frac{1}{2}\right) &= \frac{1}{2} > 1 - \frac{s_i' + \frac{1}{2}}{2} = u_i\left(s_i', s_j = \frac{1}{2}\right), \text{ where } s_i' > \frac{1}{2} \end{aligned}$$

Thus, we conclude that the unique pure-strategy NE is $(s_1 = 1/2, s_2 = 1/2)$.

1.2 Three vendors

We consider all cases of pure strategy profiles and show that in each case at least one player has an incentive to deviate.

First, suppose the players choose three different locations, say $s_1 < s_2 < s_3$. It is easy to check that each player has a profitable deviation. For example, player 1 has an incentive to choose $s_1' \in (s_1, s_2)$.

Now, suppose that two players, say 1 and 2 choose the same location s and player 3 chooses $s_3 \neq s$. If $s_3 > s$ player 3 prefers to choose any $s_3' \in (s, s_3)$, while if $s_3 < s$ player 3 prefers to choose any $s_3' \in (s_3, s)$.

Finally, suppose all players choose the same location s. The payoff of every player is $1/3$. Suppose that $s \neq 1/2$. Then, player 1 can choose $s_1 = 1/2$ and assure herself of a payoff greater than $1/2$, which is greater than $1/3$. Hence, player 1 has an incentive to deviate. Suppose instead that $s = 1/2$. Then, there exists an $\varepsilon > 0$ sufficiently small such that

$$u_1 \left(s_1 = 1/2 - \varepsilon, s_2 = 1/2, s_3 = 1/2\right) > \frac{1}{3}.$$

2 Air strike

We have the following normal-form game. The set of players is $\{A,\ B\}$. The sets of actions (pure strategies) are $S_A = S_B = \{1, 2, 3\}$. The players' payoffs are described in the following matrix:

	1	2	3
1	$0, 0$	$v_1, -v_1$	$v_1, -v_1$
2	$v_2, -v_2$	$0, 0$	$v_2, -v_2$
3	$v_3, -v_3$	$v_3, -v_3$	$0, 0$

where the total (non-destroyed) value of the three targets for player B is normalized to zero.

Clearly, this game does not admit any pure-strategy NE. Player B would like to choose the same target as player A, while player A is better off when they choose different targets. Hence, we have to look for mixed-strategy equilibria. There are four possible cases to consider: (i) A randomizes between target 1 and target 3, but does not assign positive probability to target 2; (ii) A randomizes between target 2 and target 3; (iii) A randomizes between targets target 1 and target 2; (iv) A randomizes between all three targets.

Some of these cases can be eliminated without any computational work. First note that in an NE, if player A assigns zero probability to one of the targets, then player B has to assign zero probability to the same target. (Think about why.) This, in turn, implies that there can be no NE in which player A assigns positive probability to target 3 and zero probability to one of the other targets. To see this, suppose that there is a NE in which player A randomizes between target 2 and target 3 with probabilities p_2 and $p_3 = (1 - p_2)$. Then, it must be that player B assigns zero probability to defending target 1. However, in that case, assigning probability p_3 to target 1 is a strictly profitable deviation for player A, which is a contradiction. Hence, there cannot be a NE in the form of case (ii). In a similar way, we can show that there cannot be a NE in the form of case (i). Therefore, we are left with cases (iii) and (iv).

(Case iii) Let the strategy of player A be $\sigma_A = (\alpha, 1 - \alpha, 0)$ where $\alpha \in (0, 1)$. As argued above, player B will not defend target 3. That is, player B will use a strategy on the form $\sigma_B = (\gamma, 1 - \gamma, 0)$ where $\gamma \in [0, 1]$.

If $(\alpha, 1 - \alpha, 0)$ is an equilibrium strategy for player A, it must be the case that:

$$u_A(1,(\gamma,1-\gamma,0)) = (1-\gamma)v_1 = \gamma v_2 = u_A(2,(\gamma,1-\gamma,0)),$$
$$u_A(1,(\gamma,1-\gamma,0)) = (1-\gamma)v_1 \geqslant v_3 = u_A(3,(\gamma,1-\gamma,0)),$$

which implies:

$$\gamma = \tfrac{v_1}{v_1+v_2},$$
$$\tfrac{v_1 v_2}{v_1+v_2} \geqslant v_3.$$

Since $\gamma = \frac{v_1}{v_1+v_2} \in (0,1)$, it must be the case that player B is indifferent between target 1 and target 2, and prefers target 1 and 2 to target 3. Thus, we have:

$$u_B((\alpha,1-\alpha,0),1) = -(1-\alpha)v_2 = -\alpha v_1 = u_B((\alpha,1-\alpha,0),2),$$
$$u_B((\alpha,1-\alpha,0),1) = -(1-\alpha)v_2 \geqslant -\alpha v_1 - (1-\alpha)v_2 = u_B((\alpha,1-\alpha,0),3)$$

which implies:

$$\alpha = \frac{v_2}{v_1+v_2}$$

where we omit the inequality, after noticing that it is always satisfied. Thus, if $\frac{v_1 v_2}{v_1+v_2} \geqslant v_3$ the strategy profile $\sigma_A = \left(\frac{v_2}{v_1+v_2}, \frac{v_1}{v_1+v_2}, 0\right)$, $\sigma_B = \left(\frac{v_1}{v_1+v_2}, \frac{v_2}{v_1+v_2}, 0\right)$ constitutes a Nash equilibrium.

(Case iv) Let the strategy of player A be $\sigma_A(1) = (\alpha, \beta, 1-\alpha-\beta)$ where $\alpha \in (0,1)$, $\beta \in (0,1)$ and $1-\alpha-\beta > 0$. Let $(\gamma, \delta, 1-\gamma-\delta)$ denote the strategy of player B. Player A is willing to use all actions if they all yield the same expected payoff:

$$u_A(1,(\gamma,\delta,1-\gamma-\delta)) = (1-\gamma)v_1 = (1-\delta)v_2 = u_A(2,(\gamma,\delta,1-\gamma-\delta)),$$
$$u_A(1,(\gamma,\delta,1-\gamma-\delta)) = (1-\gamma)v_1 = (\gamma+\delta)v_3 = u_A(3,(\gamma,\delta,1-\gamma-\delta)).$$

The solution to the above system is:

$$\gamma^* = \tfrac{v_1 v_2 + v_1 v_3 - v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}$$
$$\delta^* = \tfrac{v_1 v_2 - v_1 v_3 + v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}$$
$$1-\gamma-\delta = \tfrac{-v_1 v_2 + v_1 v_3 + v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}$$

Notice that $\gamma > 0$ and $\delta > 0$. Of course, we also need $1-\gamma-\delta \geqslant 0$. This holds if and only if:

$$v_3 \geqslant \frac{v_1 v_2}{v_1+v_2}.$$

We now need to compute player A's equilibrium strategy.

Let us assume that $v_3 > \frac{v_1 v_2}{v_1+v_2}$. In this case, player B assigns positive probability to all the actions. Thus, we have to find values of α and β such that player B is indifferent among all actions:

$$
\begin{gathered}
u_B\left((\alpha,\beta,1-\alpha-\beta),1\right) = -\beta v_2 - (1-\alpha-\beta)\, v_3 = \\
-\alpha v_1 - (1-\alpha-\beta)\, v_3 = u_B\left((\alpha,\beta,1-\alpha-\beta),2\right), \\
u_B\left((\alpha,\beta,1-\alpha-\beta),1\right) = -\beta v_2 - (1-\alpha-\beta)\, v_3 = \\
-\alpha v_1 - \beta v_2 = u_B\left((\alpha,\beta,1-\alpha-\beta),3\right),
\end{gathered}
$$

which implies

$$
\begin{aligned}
\alpha^* &= \tfrac{v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3} \\
\beta^* &= \tfrac{v_1 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}.
\end{aligned}
$$

We conclude that if $v_3 > \frac{v_1 v_2}{v_1+v_2}$, then the Nash equilibrium of the game is $\sigma_A = (\alpha^*, \beta^*, 1-\alpha^*-\beta^*)$, $\sigma_B = (\gamma^*, \delta^*, 1-\gamma^*-\delta^*)$.

If $v_3 < \frac{v_1 v_2}{v_1+v_2}$, then player B assigns positive probability only to target 1 and target 2. Thus, we have:

$$
\begin{gathered}
u_B\left((\alpha,\beta,1-\alpha-\beta),1\right) = -\beta v_2 - (1-\alpha-\beta)\, v_3 = \\
-\alpha v_1 - (1-\alpha-\beta)\, v_3 = u_B\left((\alpha,\beta,1-\alpha-\beta),2\right), \\
u_B\left((\alpha,\beta,1-\alpha-\beta),1\right) = -\beta v_2 - (1-\alpha-\beta)\, v_3 \geqslant \\
-\alpha v_1 - \beta v_2 = u_B\left((\alpha,\beta,1-\alpha-\beta),3\right),
\end{gathered}
$$

which, in turn, implies:

$$
\begin{gathered}
\alpha \geqslant \tfrac{v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3} = \tfrac{v_2}{2(v_1+v_2)}, \\
\beta = \tfrac{v_1}{v_2}\alpha.
\end{gathered}
$$

To sum up, we have the following cases.

- If $v_3 < \frac{v_1 v_2}{v_1+v_2}$, there exists a unique Nash equilibrium

$$
\begin{aligned}
\sigma_A &= \left(\tfrac{v_2}{v_1+v_2}, \tfrac{v_1}{v_1+v_2}, 0\right) \\
\sigma_B &= \left(\tfrac{v_1}{v_1+v_2}, \tfrac{v_2}{v_1+v_2}, 0\right)
\end{aligned}
$$

- If $v_3 > \frac{v_1 v_2}{v_1+v_2}$, there exists a unique Nash equilibrium

$$
\begin{aligned}
\sigma_A &= \left(\tfrac{v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, \tfrac{v_1 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, \tfrac{v_1 v_2}{v_1 v_2 + v_1 v_3 + v_2 v_3}\right) \\
\sigma_B &= \left(\tfrac{v_1 v_2 + v_1 v_3 - v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, \tfrac{v_1 v_2 - v_1 v_3 + v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}, \tfrac{-v_1 v_2 + v_1 v_3 + v_2 v_3}{v_1 v_2 + v_1 v_3 + v_2 v_3}\right)
\end{aligned}
$$

- If $v_3 = \frac{v_1 v_2}{v_1+v_2}$. there exist a continuum of Nash equilibria:

$$
\begin{aligned}
\sigma_A &= \left(\alpha, \tfrac{v_1}{v_2}\alpha, 1 - \left(\tfrac{v_1+v_2}{v_2}\right)\alpha\right) \\
\sigma_B &= \left(\tfrac{v_1}{v_1+v_2}, \tfrac{v_2}{v_1+v_2}, 0\right)
\end{aligned}
$$

where $\alpha \in \left[\frac{v_2}{2(v_1+v_2)}, \frac{v_2}{(v_1+v_2)}\right]$.

Clearly, as v_3 goes to zero we have only the Nash equilibrium in which both player A and player B randomize between target 1 and target 2. This is very intuitive. When the value of target 3 is negligible, player A does not have an incentive to attack it. Since player A does not attack target 3, player B does not defend it.

3 First-Price auction with different valuations

First note that, in equilibrium, each player $i = 1, \ldots, n$ can not get the object by bidding $b_i > v_i$. In fact, when $b_i > v_i$ and player i wins the auction, her payoff is negative $(v_i - b_i)$. But then, player i would have an incentive to deviate and bid, for example, zero.

Now, suppose that player 1 does not get the object. Player 1's payoff is zero. Moreover, we just argued that the highest bid then has to be smaller than or equal to v_2. But since $v_1 > v_2$, a profitable deviation for player 1 is to choose a bid in the interval (v_2, v_1) and get a positive payoff.

Notice that this game admits many pure-strategy Nash equilibria. Any combination of bids on the following form

$$b_1 = b, b_2 \leqslant b, \ldots, b_n \leqslant b$$

with $b \in [v_2, v_1]$ and $b_j = b$ for at least one player $j \in \{2, \ldots, n\}$, is an equilibrium.

4 A simple Bayesian game

$t_1 = a$

	L	R
U	$2, 2$	$-2, 0$
D	$0, -2$	$0, 0$

$t_1 = b$

	L	R
U	$0, 2$	$1, 0$
D	$1, -2$	$2, 0$

First of all notice that in any BNE, player 1 chooses D when her type is b. In fact, for any action of player 2, the payoff of type b (of player 1) from action D is strictly greater than the payoff from action U.

Suppose that player 2 chooses L. Then type a (of player 1) chooses U and type b chooses D. The expected payoff of player 2 is $.9\,(2) + .1\,(-2) = 1.6$. Notice that if player 2 plays R her (expected) payoff is 0. So the strategy profile $((U, D),\ L)$ constitutes a BNE.

Suppose that player 2 chooses R. Then both type a and type b choose D. The (expected) payoff of player 2 is 0. If player 2 plays L, her (expected) payoff is -2. Thus we have another BNE: $((D, D),\ R)$.

Finally, suppose that player 2 randomizes between L and R. Let β denote the probability that player 2 chooses L. Let α denote the probability that type

a chooses U. Remember that type b chooses D in any BNE. Player 2 is willing to randomize if and only if:

$$.9\,(2\alpha - 2\,(1-\alpha)) + .1\,(-2) = 0$$

which implies $\alpha = \frac{5}{9}$. Type a of player 1 is willing to randomize if and only if:

$$2\beta - 2\,(1-\beta) = 0$$

which implies $\beta = \frac{1}{2}$.

To summarize, the game has the following BNE: $((U, D)\,,\ L)$, $((D, D)\,,\ R)$ and $((\alpha = 5/9, D)\,,\ \beta = 1/2)\,.$

5 An exchange game

The sets of types of the two players are $T_1 = T_2 = \{x_1, \ldots, x_n\}$. The distribution function over types for each player is F. However, the distribution will not matter for the argument made below. (Since the set of types is *finite*, we make the standard assumption that all types have positive probability. If a type has zero probability, we could simply delete it and the analysis below goes through).

Each player $i \in \{1,\ 2\}$ has the same action set $A_i = \{Y, N\}$, where Y (N) means that a player is (is not) willing to exchange the prizes. A pure strategy is a function from types to actions, $s_i : T_i \to A_i$. Finally, the payoffs, as functions of actions and types, are given by:

$$u_i\,(a_i, a_j, x_i, x_j) = \begin{cases} x_j & \text{if } a_i = a_j = Y \\ x_i & \text{otherwise.} \end{cases}$$

[Note that we could instead express the payoffs as functions of strategies (and types), though to keep notation simple we express them as functions of actions.]

Now, suppose that a player, say 1, is willing to exchange at a prize $x_i > x_1$. That is, suppose that $\sigma_1\,(Y|x_i) > 0$ for some $x_i > x_1$. This implies that player 2 has to exchange when she has type x_1. That is, $\sigma_2\,(Y|x_1) = 1$. Now consider type x_i of player 1. If she does not exchange, her payoff is x_i. On the other hand, if she exchanges there is a positive probability that her payoff is smaller than x_i (player 1 could face type x_1 of player 2 who always trades). Therefore, type x_i of player 1 is willing to exchange only if there exists a type $x_j > x_i$ of player 2 who trades with positive probability. Now, consider type x_j of player 2. Type x_j of player 2 is willing to trade only if there exists a type $x_k > x_j$ of player 1 who trades with positive probability. By applying this argument a finite number of times (remember that the set of types is finite) we conclude that there exists a player, say 2, who trades with positive probability when her type is x_n (the highest possible type). Obviously, trading is not optimal for type x_n : with positive probability she receives a payoff smaller than x_n, while if she does not trade her payoff is x_n. Hence, we have a contradiction.