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Game Theory

Solutions to Problem Set 6

1 Mixed and behavioral strategies

(a) The behavioral strategy equivalent to σ_1^* is: $\beta(A) = 0.5$, $\beta(l) = 0.4$. This follows from:

$$\begin{aligned}
b_1(A|h_1) &= \sigma_1(A,l) + \sigma_1(A,r) = \frac{1}{2} \\
b_1(B|h_1) &= \sigma_1(B,l) + \sigma_1(B,r) = \frac{1}{2} \\
b_1(l|h_3) &= \frac{\sigma_1(A,l)}{b_1(A|h_1)} = \frac{0.2}{0.5} = 0.4 \\
b_1(r|h_3) &= \frac{\sigma_1(A,r)}{b_1(A|h_1)} = \frac{0.3}{0.5} = 0.6
\end{aligned}$$

Note that, since playing B ends the game, the (l,r) information set h_1 is not reached after B. Hence, any probability distribution such that $\sigma_1(B,l) + \sigma_1(B,r) = b_1(B|h_1) = \frac{1}{2}$ and $\sigma_1(A,l) = 0.2$, $\sigma_1(A,r) = 0.3$ will describe a mixed strategy that is equivalent to σ_1^*.

(b) Notice that at the information set h_1

$$b_1(B|h_1) = \sigma_1(B,l) + \sigma_1(B,r) = 1.$$

Thus, information set h_3 is never reached, and (l,r) are never played. Therefore, any behavioral strategy that assigns $b_1(B|h_1) = 1$ and $b_1(l|h_3) = q, b_1(r|h_3) = 1-q$ for any $q \in [0,1]$ is equivalent to the mixed strategy given in the question. This implies that there is no unique behavioral strategy representation.

2 Nash equilibria of extensive-form game

(a) The sets of (pure) strategies are:

$$\begin{aligned}
S_1 &= \{AUU, AUD, ADU, ADD, BUU, BUD, BDU, BDD\} \\
S_2 &= \{L, R\}.
\end{aligned}$$

Now, define the mixed strategies:

$$\begin{array}{l}
\sigma_1^*(AUU) = \sigma_1^*(AUD) = \frac{p}{2} \\
\sigma_1^*(ADU) = \sigma_1^*(ADD) = \frac{q}{2} \\
\sigma_1^*(BUU) = \sigma_1^*(BDU) = \frac{r}{2} \\
\sigma_1^*(BUD) = \sigma_1^*(BDD) = \frac{1-p-q-r}{2} \\
\sigma_2^*(L) = x \\
\sigma_2^*(R) = 1 - x.
\end{array}$$

Note that we here imposed that payoff equivalent strategies are played with the same probability. This is done only to simplify the calculations below. (Note that we are only looking for *one* NE, not all of the NE.)

Player 2' s payoff are:

$$\begin{aligned}
EU_2(L|\sigma_1^*) &= 2q + 3r \\
EU_2(R|\sigma_1^*) &= p + 2r
\end{aligned}$$

For player 2 to mix, she has to be indifferent between playing L and R:

$$EU_2(L|\sigma_1^*) = EU_2(R|\sigma_1^*) \Longrightarrow 2q = p - r \tag{1}$$

Player 1's payoffs are:

$$\begin{aligned}
EU_1(AUU, \sigma_2^*) &= EU_1(AUD, \sigma_2^*) = x \\
EU_1(ADU, \sigma_2^*) &= EU_1(ADD, \sigma_2^* = 2 - 3x \\
EU_1(BUU, \sigma_2^*) &= EU_1(BDU, \sigma_2^*) = 7x - 3 \\
EU_1(BUD, \sigma_2^*) &= EU_1(BDD, \sigma_2^*) = x
\end{aligned}$$

For player 1 to mix (and assign positive probability to *all* pure strategies), the following equalities have to hold:

$$EU_1(AUU, \sigma_2^*) = EU_1(ADU, \sigma_2^*) = EU_1(BUU, \sigma_2^*) = EU_1(BUD, \sigma_2^*) \Rightarrow x = \frac{1}{2} \tag{2}$$

Any mixed strategy that satisfy both equation (1) and (2) is a mixed strategy NE. For instance, the following mixed strategy profile constitute a NE:

$$\begin{array}{l}
\sigma_1^*(AUU) = \sigma_1^*(AUD) = 0.2 \\
\sigma_1^*(ADU) = \sigma_1^*(ADD) = 0.05 \\
\sigma_1^*(BUU) = \sigma_1^*(BDU) = 0.1 \\
\sigma_1^*(BUD) = \sigma_1^*(BDD) = 0.15 \\
\sigma_2^*(L) = 0.5 \\
\sigma_2^*(R) = 0.5
\end{array}$$

(b) The equivalent behavioral strategies are:

$$\beta_1(A) = 0.5, \quad \beta_1(U|A) = 0.8, \quad \beta_1(U|B) = 0.4, \quad \beta_2(L) = 0.5.$$

3 A parlor game

The sets of (pure) strategies for the two players are:

$$\begin{aligned} S_1 &= \{L_H L_L,\ L_H H_L,\ H_H L_L,\ H_H H_L\} \\ S_2 &= \{C,\ L\} \end{aligned}$$

where A_B indicates that type B of player 1 plays A.

Now, notice that if the H card is realized then player 1 will play H_H with probability 1 (that is, type H will play H_H), since any strategy with L_H is strictly dominated by a strategy where H_H is played. Then, define the following:

$$\sigma_1^*\left(L_H L_L,\ L_H H_L,\ H_H L_L,\ H_H H_L\right) = (0,\ 0,\ \ p,\ 1-p)$$
$$\sigma_2^*\left(C, I\right) = (q, 1-q)$$

Given σ_1^*, σ_2^*, we can compute the expected payoff of each action:

$$\begin{aligned} EU_1(L_L|\sigma_2^*) &= -1 \\ EU_1(H_L|\sigma_2^*) &= q + (1-q)(-4) = 5q - 4 \\ EU_2(C|\sigma_1^*) &= \frac{1}{2}(-1) + \frac{1}{2}[(1-p) - p] = -p \\ EU_2(I|\sigma_1^*) &= \frac{1}{2}(-4) + \frac{1}{2}[(1-p) + 4p] = \frac{3}{2}(p-1) \end{aligned}$$

For player 1 of type L:

$$BR_1^L(\sigma_2^*) = \begin{cases} p = 1 \text{ if } q > 0.6 \\ p \in [0,1] \text{ if } q = 0.6 \\ p = 0 \text{ if } q < 0.6 \end{cases}$$

For player 2:

$$BR_2(\sigma_1^*) = \begin{cases} q = 0 \text{ if } p > 0.6 \\ q \in [0,1] \text{ if } p = 0.6 \\ q = 1 \text{ if } p < 0.6 \end{cases}$$

Hence, the NE is:

$$\begin{aligned} \sigma_1^*\left(L_H L_L,\ L_H H_L,\ H_H L_L,\ H_H H_L\right) &= (0,\ 0,\ \ .4,\ .6) \\ \sigma_2^*\left(C, I\right) &= (.6,\ .4). \end{aligned}$$

In words, player 1 plays H when she is type H, and plays H with probability .6 when she is type L. Player 2 plays C with probability .6 if 2's information set is reached.