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Game Theory

Solutions to Problem Set 7

1 The Centipede Game.

(a) The centipede game does admit NE which are not subgame-perfect. Consider the following strategies: player 2 always exits, player 1 exits in the first information set, and thereafter continues. This is a NE. Given player 2's strategy, the BR of player 1 is to exit at the first information set. Given player1's strategy, player 2 can not do better by changing her strategy.
It is also easy to check that any strategy profile in which player 1 exits in her first information set and player 2 exits with probability $\beta \geqslant 2/3$ in her first information set constitutes a NE.

(b) All NE have the same outcome. To see that, suppose there is a NE in which player 1 continues at the first move with some positive probability. This will be true only if player 2 is playing "continue" in the following move with a strictly positive probability. However, for player 2 to play continue with positive probability, it has to be the case that player 1 is playing "continue" at the third node with a strictly positive probability. The argument continues to the last move of the game, where again one need to assume positive probability (approaching one) of player 2 to continue. However, this does not constitute a NE, since player 2 has a profitable deviation (exiting) at the last node.

2 Sequential Bargaining.

Suppose there exists a SPE of the entire game where the allocation of a dollar is according to $(s, 1-s)$, where s share is allocated to player 1.

Thus expecting a SPE $(s, 1-s)$ in (any subgame of) period 3, player 1 will accept any offer of

$$s_2 \geq \delta_1 s$$

Player 2, knowing this, will then offer

$$\begin{aligned} s_2^* &= \delta_1 s \\ 1 - s_2^* &= 1 - \delta_1 s > \delta_2(1-s) \end{aligned}$$

Expecting s_2, in period 1, player 2 would accept any offer such that:

$$\begin{aligned}(1-s_1) &\geq \delta_2(1-s_2^*)\\ &= \delta_2(1-\delta_1 s)\end{aligned}$$

Thus, player 1 will offer at period one:

$$\begin{aligned}s_1^* &= 1-\delta_2(1-\delta_1 s) > \delta_1 s_2^* = \delta_1^2 s\\ 1-s_1^* &= \delta_2(1-\delta_1 s)\end{aligned}$$

Now, let s_H be the highest share for player 1 in any SPE, and let s_L be the lowest.

Since the offer at period 1, s_1^*, is increasing in s, the highest share for player 1 at period one would be:

$$s_1^*(s_H) = 1-\delta_2(1-\delta_1 s_H).$$

where we note that $s_1^*(s_H)$ is increasing in s_H. Since s_H is the highest SPE share for player one in the entire game, it must be the case that:

$$\begin{aligned}s_H &= s_1^*(s_H) = 1-\delta_2(1-\delta_1 s_H)\\ s_H &= \frac{1-\delta_2}{1-\delta_2\delta_1}\end{aligned}$$

Similarly, the lowest share for player 1 at period one would be:

$$s_1^*(s_L) = 1-\delta_2(1-\delta_1 s_L)$$

However, since s_L is the lowest SPE share for player one in the entire game, by definition it must be the case that:

$$\begin{aligned}s_L &= s_1^*(s_L) = 1-\delta_2(1-\delta_1 s_L)\\ s_L &= \frac{1-\delta_2}{1-\delta_2\delta_1}\end{aligned}$$

Since $s_L = s_H = \dfrac{1-\delta_2}{1-\delta_2\delta_1}$, then it follows that there exist a unique SPE of the game where the settlement is:

$$(s, 1-s) = \left(\frac{1-\delta_2}{1-\delta_2\delta_1}, \frac{\delta_2\,(1-\delta_1)}{1-\delta_2\delta_1}\right)$$

3 Gibbons, Exercise 2.1

By backward induction, in the second period the parent chooses B to solve:

$$\max_B V(I_p(A) - B) + kU(I_c(A) + B)$$

for any value of A. The parents reaction function B*(A) is implicitly defined by the following FOC:

$$-V'(I_p(A) - B^*(A)) + kU'(I_c(A) + B^*(A)) = 0.$$

Since we are looking for a SPE (and not just a NE), this condition has to hold for any A. Using the implicit function theorem, we get:

$$-V''(I_p(A) - B^*(A)) \left[I_p'(A) - \frac{dB^*}{dA} \right] + kU''(I_c(A) + B^*(A)) \left[I_c'(A) + \frac{dB^*}{dA} \right] = 0$$

$$\Rightarrow \frac{dB^*}{dA} = -\frac{kU'' * I_c'(A) - V'' * I_p'(A)}{k * U'' + V''} \tag{1}$$

In the first period, the child, in a SPE, chooses A to solve:

$$\max_A U(I_C(A) + B^*(A))$$

The equilibrium value of A follows from the FOC:

$$U'(I_C(A) + B^*(A)) \left[I_c'(A) + \frac{dB^*}{dA} \right] = 0$$

which, since U' is always greater than zero, reduces to:

$$\left[I_c'(A) + \frac{dB^*}{dA} \right] = 0 \tag{2}$$

Substituting (1) into (2) we get:

$$I_c'(A) - \frac{kU'' * I_c'(A) - V'' * I_p'(A)}{k * U'' + V''} = 0$$

which simplifies to:

$$V'' * \left[I_c'(A) + I_p'(A) \right] = 0.$$

Now, since V(·) is strictly concave, V"<0. Hence, the condition reduces to:

$$\left[I_c'(A) + I_p'(A) \right] = 0.$$

Finally, noting that both $I_c(A)$ and $I_p(A)$ are strictly concave, we conclude that the child is choosing A to maximize $[I_c(A) + I_p(A)]$. That is, the child, in equilibrium, chooses a value of A that maximizes the joint income.

4 Gibbons, Exercise 2.4

We solve this game using backwards induction. Starting with the final stage, observing C_1 the second partner will set:

$$C_2 = \begin{cases} R - C_1, \text{ if } (R - C_1)^2 \leq V \ \text{ and } R - C_1 \geq 0 \\ 0, \text{ otherwise} \end{cases}$$

Thus, anticipating the reaction function of the second partner, the first one will choose:

$$C_1 = \begin{cases} 0, \text{ if } R \leq \sqrt{V} \text{ and } \delta V \geq V - R^2 \\ R, \text{ if } R \leq \sqrt{V} \text{ and } \delta V < V - R^2 \\ 0, \text{ if } R > \sqrt{V} \text{ and } \delta V - (R - \sqrt{V})^2 < 0 \\ R - \sqrt{V}, \text{ if } R > \sqrt{V} \text{ and } \delta V - (R - \sqrt{V})^2 \geq 0 \end{cases}$$

The equilibrium outcome is:

$$(C_1, C_2) = \begin{cases} (0, R) & \text{if } R \leq \sqrt{V} \text{ and } & \delta V \geq V - R^2 \\ (R, 0) & \text{if } R \leq \sqrt{V} \text{ and } & \delta V < V - R^2 \\ (R - \sqrt{V}, \ \sqrt{V}) & \text{if } R > \sqrt{V} \text{ and } & \delta V - (R - \sqrt{V})^2 \geq 0 \\ (0, 0) & \text{if } R > \sqrt{V} \text{ and } & \delta V - (R - \sqrt{V})^2 < 0 \end{cases}$$

5 Gibbons, Exercise 2.5

5.1 Job Assignment

Using backwards induction, we have that in period 3 the firm will make the following choice if the worker has invested in skill-training:

$$\begin{cases} D, \text{ if } \ (y_{DS-}w_D) \geq (y_{ES} - w_E) \Leftrightarrow (y_{DS-}y_{ES}) \geq w_D - w_E \\ E, \text{ otherwise} \end{cases}$$

If the worker has not invested in skill, assuming $(w_D - w_E) \geq 0$, the firm will always (i.e. for any pair $w_D, \ w_E$) choose to assign the worker to the easy job E. To see this, note that:

$$(y_{D0-}w_D) < (y_{E0} - w_E) \Leftrightarrow (y_{D0-}y_{E0}) < (w_D - w_E)$$

which always holds, since $y_{D0-}y_{E0} < 0$. Note also that in any SPE it must in fact be the case that $(w_D - w_E) \geq 0$ on the equilibrium path.

5.2 Skill Investment Decision

In period 2 the optimal behavior of the worker, who anticipates the behavior of the firm, depends on whether (i) $y_{DS-}y_{ES} \geq w_D - w_E$, or (ii) $y_{DS-}y_{ES} < w_D - w_E$. We analyze these cases separately.

Case (i): $y_{DS-}y_{ES} \geq w_D - w_E$. The employer will choose D in the final round, if the worker invests. This gives the worker a payoff of $w_D - c$. The employer will choose E in the final round, if the worker does not invest. This gives the worker a payoff of w_E. Hence, the optimal decision of the worker is:

$$\begin{cases} S, \text{ if } \ (w_D - c) \geq w_E \\ NS, \text{ if } \ (w_D - c) < w_E \end{cases}$$

Case (ii): $y_{DS-}y_{ES} < w_D - w_E$. In this case, the worker knows that the employer will choose E in the final round, no matter what the worker does. Hence, not investing in skill (NS) is optimal for the worker.

Summarizing case (i) and (ii) we have that the optimal choice of the worker is:

$$\begin{cases} S, \text{ if } \ (w_D - c) \geq w_E \quad and \quad (y_{DS-}y_{ES}) \geq (w_D - w_E) \\ NS, \text{ otherwise} \end{cases}$$

5.3 Wage Schedule and SPE Outcome

In the first period, the firm chooses a wage schedule. That is, it chooses w_D and w_E, and it does so with the objective of maximizing its profits..

Note first that since the outside option for the worker is earning zero, the firm sets w$_E$ = 0. Hence, all we need to do is to find the w$_D$ that is optimal from the firms point of view. Since $y_{DS} - y_{E0} > c$, it is efficient for the firm to have the worker investing in skill and later be assigned to the difficult task. However, assigning the worker to the difficult task is a credible promise only if $y_{DS} - y_{ES} \geq w_D - w_E$. In addition, we know what the worker would invest in skill only if $w_D - w_E \geq c$. Thus, we have two cases:

Case (1): $y_{DS} - y_{ES} \geq c$. In this case, the firm will set $w_D - w_E = c$. That is, it will set $w_D = c$, $w_E = 0$, and we have the following SPE outcome:

$$\begin{aligned} Period\ 1 &: \text{ Firm sets } (w_D = c\ w_E = 0) \\ Period\ 2 &: \text{ Worker chooses S (invest)} \\ Period\ 3 &: \text{ Firm assigns worker to D} \end{aligned}$$

Case (2): $y_{DS} - y_{ES} < c$. In this case there is no credible way for the firm to induce investment on the part of the worker. Hence, in this case the outcome will be such that the the worker chooses not to invest (NS) and the firm assigns the worker to the easy (E) job. What the firm does in the first period is irrelevant. For instance, the firm can set w_D, w_E, such that its offer to assign the worker to the difficult task is credible, but the worker finds it unprofitable to invest in acquiring the skill. In this case the outcome is such that:

$$\begin{aligned}
\textit{Period } 1 &: \text{ Firm sets } (w_D \in [0,\ y_{DS} - y_{ES}],\ w_E = 0) \\
\textit{Period } 2 &: \text{ Worker chooses NS (not invest)} \\
\textit{Period } 3 &: \text{ Firm assigns worker to E}
\end{aligned}$$

Another possibility is for the firm to set the wages in such a way that its offer to assign the worker to the difficult task is not credible. In this case, we have the following outcome:

$$\begin{aligned}
\textit{Period } 1 &: \text{ Firm sets } (w_D \in (y_{DS} - y_{ES},\ \infty),\ w_E = 0) \\
\textit{Period } 2 &: \text{ Worker chooses NS (not invest)} \\
\textit{Period } 3 &: \text{ Firm assigns worker to E}
\end{aligned}$$