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Game Theory
Solutions to Problem Set 8

Question 1

	A	B	C
A	5, 5	0, 6	0, 0
B	6, 0	3, 3	0, 0
C	0, 0	0, 0	1, 1

The stage game G admits two pure-strategy Nash equilibria, (B, B) and (C, C), and a mixed-strategy Nash equilibrium (σ_1^*, σ_2^*) in which each player plays B with probability $1/4$ and C with probability $3/4$. (To verify that there are no other NE, notice that action A is strictly dominated). The stage game equilibrium payoffs are:

$$
\begin{aligned}
g_1(B, B) &= g_1(B, B) = 3 \\
g_1(C, C) &= g_1(C, C) = 1 \\
g_1(\sigma_1, \sigma_2) &= g_1(\sigma_1, \sigma_2) = 3/4.
\end{aligned}
$$

If we restict ourselves to looking only at **pure strategies**, it is straight-forward to check that the highest and lowest average SPE payoffs in symmetric strategies are:

$$
\begin{aligned}
\bar{u}(T) &= \frac{5(T-1)+3}{T} \\
\underline{u}(T) &= 1
\end{aligned}
$$

If we allow for **mixed strategies**, the problem becomes somewhat more difficult. First note that in any SPE of $G(T)$, the players have to play a NE of G in the last period. Moreover, there is no action profile in G (the stage game) that gives a payoff greater than 5 to both players. Therefore, in a symmetric SPE of $G(T)$, the players can get at most a payoff of 5 in any period $t = 0, \ldots, T-1$, and at most a payoff of 3 in period T. We now show that there exists a symmetric SPE that allows the players to achieve these payoffs. Consider the following symmetric strategy profile:

$$
s_i\left(h^t\right) = \begin{cases} A & \text{if } h^t = h^0 \text{ or } t < T \text{ and } h^t = ((A, A), \ldots, (A, A)) \\ B & \text{if } t = T \text{ and } h^{T-1} = ((A, A), \ldots, (A, A)) \\ C & \text{otherwise} \end{cases}
$$

It is easy to check that (s_1, s_2) is a SPE. Consider a history $h^t = ((A, A), \ldots, (A, A))$ where $t < T$. By following the equilibrium strategy a player gets a continuation payoff equal to $5(T-1-t)+3$. By deviating, the player can get at most $6 + (T-1-t)$. Clearly, the deviation is not profitable. After history $h^{T-1} = ((A, A), \ldots, (A, A))$ the players are playing a NE of G, hence do not have incentives to deviate. Similarly, after any history $h^t \neq ((A, A), \ldots, (A, A))$ the players always play the same NE of G, so no profitable deviations exist. Therefore, we have:

$$\bar{u}(T) = u_1(s_1, s_2) = u_2(s_1, s_2) = \frac{5(T-1)+3}{T}.$$

In order to find $\underline{u}(T)$, we first compute the minmax values. Look at player 1 (both players have the same minmax value since the game is symmetric). Suppose player 2 chooses action B with probability $x \in [0, 1]$ and action C with probability $1-x$. If $x \leqslant 1/4$, player 1 will choose C and her payoff will be $(1-x) \geqslant 3/4$. If $x \geqslant 1/4$, then player 1 will choose B and her payoff be $3x \geqslant 3/4$. Therefore, the smallest payoff that player 1 can get when she behaves optimally and player 2 randomizes between B and C is equal to $3/4$. In a similar way, it is easy to check that if player 2 randomizes between A and C (or A and B), then the smallest payoff that player 1 can get (when she best responds) is equal to $6/7$. (Check this.) Finally, it is also easy to verify that player 1 can assure herself a payoff greater than $3/4$ if player 2 randomizes between A, B and C. To see this, suppose that player 2 chooses actions A, B and C with probabilities x, y and $1-x-y$, respectively. Suppose that player 1 chooses B if $x+y \geqslant 1/4$ and C if $x+y < 1/4$ (this is not the best response of player 1, but sufficient to make the case). With this strategy player 1 can guarantee herself a payoff greater than or equal to $3/4$. We conclude that the minmax value is $3/4$, for both players.
Notice that $3/4$ is also a NE payoff of the stage game. Consider the symmetric strategy profile (s'_1, s'_2) of the repeated game in which player i ($i = 1, 2$) plays the behavioral strategy σ_i^* (defined above) after any history. Of course, (s'_1, s'_2) is a SPE of $G(T)$. Since the payoff of a player in any Nash equilibrium of a game cannot be smaller than her minmax value, we conclude that:

$$\underline{u}(T) = u_1(s'_1, s'_2) = u_2(s'_1, s'_2) = \frac{3}{4}.$$

Question 2

	A	B	C
A	$0,0$	$3,4$	$6,0$
B	$4,3$	$0,0$	$0,0$
C	$0,6$	$0,0$	$5,5$

The stage game has two pure-strategy Nash equilibria, (A, B) and (B, A), and a mixed-strategy equilibrium (σ_1^*, σ_2^*) in which each player plays A with probability $3/7$ and B with

probability 4/7. The mixed-strategy equilibrium payoffs are: $g_1(\sigma_1^*, \sigma_2^*) = g_2(\sigma_1^*, \sigma_2^*) = 12/7$. (Notice that action C is strictly dominated.)
Consider the following strategy profile (s_1, s_2):

$$s_i(h^t) = \begin{cases} C & \text{if } h^t = h^0 \\ A & \text{if } a^0 = (C, C) \\ \sigma_i^* & \text{otherwise} \end{cases}$$

$$s_i(h^t) = \begin{cases} C & \text{if } h^t = h^0 \\ B & \text{if } a^0 = (C, C) \\ \sigma_i^* & \text{otherwise} \end{cases}$$

where σ_i^* is the behavioral strategy that assigns probability 3/7 to action A and probability 4/7 to action B. In every subgame of the final period, the players play a Nash equilibrium of the original game. So we need to check the first-period incentives.
Consider player 2: If player 2 follows the equilibrium strategy, her payoff is equal to $5 + \delta 4$. The largest payoff that she can get if she deviates is equal to $6 + \delta \frac{12}{7}$. Player 2 follows the equilibrium strategy if and only if $\delta \geqslant 7/16$.
Consider player 1 : If player 1 follows the equilibrium strategy her payoff is $5 + \delta 3$. If she deviates she will get at most $6 + \delta \frac{12}{7}$. Player 1 does not deviate if and only if $\delta \geqslant 7/9$. Thus, when $\delta \geqslant 7/9$ the strategy profile (s_1, s_2) is a SPE of the repeated game.
Note that the way we set up the strategies, the necessary discount factor for player 1 is greater than from player 2. (We could have set up the strategies in such a way that the necessary discount factor for player 2 instead was the binding one.) Note also that in order to induce (C, C) in the first period, the punishment in the second period has to be the mixed-strategy Nash equilibrium (σ_1^*, σ_2^*) of the stage game.

Question 3

The stage game is:

	P_2	Q_2	R_2	S_2
P_1	$2, 2$	$x, 0$	$-1, 0$	$0, 0$
Q_1	$0, x$	$4, 4$	$-1, 0$	$0, 0$
R_1	$0, 0$	$0, 0$	$0, 2$	$0, 0$
S_1	$0, -1$	$0, -1$	$-1, -1$	$2, 0$

where $x > 4$. Consider the following given strategy profile (s_1, s_2):

$$s_i(h^t) = \begin{cases} Q_i & \text{if } h^t = 0 \\ P_i & \text{if } h^1 = (Q_1, Q_2) \text{ or } h^1 = (y, z) \text{ where } y \neq Q_1,\ z \neq Q_2 \\ R_i & \text{if } h^1 = (y, Q_2) \text{ where } y \neq Q_1 \\ S_i & \text{if } h^1 = (Q_1, z) \text{ where } z \neq Q_2 \end{cases}$$

First note that after any possible history, a stage game NE is played in the second period. Now to verify that the strategy profile (s_1, s_2) is a SPE, we need to check that there is no profitable deviation in the first period. Since the game is symmetric, it is enough to check for one of the players. Following the equilibrium strategy, player 1 receives a payoff of $4+2=6$. Of course, player 1 does not have any incentive to play R_1 or S_1 in the first period. However, if player 1 deviates to P_1 her total payoff is equal to x. Thus, (s_1, s_2) is a SPE of the repeated game if and only if $x \leq 6$.

Question 4

Consider the following strategy profile (s_1, s_2) :

$$s_1\left(h^t\right) = \begin{cases} B & \text{if } h^t = h^0 \\ T & \text{if } h_1 = (B, R) \\ M & \text{otherwise} \end{cases}$$

$$s_2\left(h^t\right) = \begin{cases} R & \text{if } h^t = h^0 \\ L & \text{if } h_1 = (B, R) \\ C & \text{otherwise} \end{cases}$$

It is easy to check that (s_1, s_2) is a SPE of the repeated game. In any subgame of the second period the players play a Nash equilibrium of the stage game. Player 2 does not have any incentive to deviate in the first period. For player 1 is concerned, if she follows the equilibrium strategy her payoff is 7. By deviating she can get at most a payoff of 6.